Applied Optimization and Game Theory
Linear Programming
Exercises and Solutions

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Exercises

Word Problems

1. Four types of hollow structural sections (HSS, a special type of metal bar or tube) are manufactured in a steel mill: small, medium, large and extra large. There are three types of machines available: A, B, and C. The following table specifies the quantity of HSS types (in terms of length, in meters) produced per hour, depending on the type of machine.

	Machine		
Hollow structural section (HSS) type	A	B	C
Small	3	6	8
Medium	2	4	7
Large	2	3	6
Extra large	1	2	3

The machines can operate 50 hours a week and the operational cost of each machine (in units of money) is: 3, 5 and 8. According to the business plan, 200, 80, 60, and 60 meters per week are needed of each of the HSS types A, B, C, and respectively D.

a) Define the above machine scheduling problem as a linear program!

b) Find an initial basis!

c) Would you solve the problem with the primal or the dual simplex algorithm? (No need to solve it!)

Solving Linear Programs: The Graphical Method

2. Consider the following linear program (Warning: There are no non-negativity constraints!):

$$\begin{array}{lrcrcl} \max & x_1 & + & x_2 & & \\ \text{s.t.} & x_1 & + & 2x_2 & \leq & 4 \\ & 2x_1 & - & x_2 & \leq & 3 \\ & x_1 & - & 2x_2 & \leq & 3 \end{array}$$

a) Solve the linear program *graphically*!

b) Solve the linear program (also graphically), when the objective function is changed to $\max -x_1 + x_2$!

c) Is there a unique optimal solution, if the objective function is $\max\ 2x_1 - x_2$? Verify your answer graphically!

3. Consider the following linear program:

$$\begin{array}{lrcrcr}
\min & -x_1 & + & 2x_2 & & \\
\text{s.t.} & x_1 & + & 2x_2 & \leq & 4 \\
 & 2x_1 & - & x_2 & \leq & 3 \\
 & x_1 & & & \geq & -1 \\
 & & & x_2 & \geq & -1
\end{array}$$

a) Illustrate the feasible region graphically!

b) Give the extreme points of the feasible region!

c) Give the optimal value of the objective function and an optimal solution!

d) Is the solution unique? Justify your answer!

e) How do the optimal value of the objective function and the corresponding solution change if we modify the objective function to $\min x_1 + x_2$? Justify your answer graphically as well!

Solving Linear Programs: The Simplex Method

4. Solve the following linear program with the simplex method:

$$\begin{array}{lrcrcrcrcr}
\max & 3x_1 & + & 8x_2 & - & 5x_3 & + & 8x_4 & & \\
\text{s.t.} & 2x_1 & + & x_2 & + & x_3 & + & 3x_4 & \leq & 7 \\
 & -x_1 & - & 2x_2 & - & x_3 & - & x_4 & \geq & -2 \\
 & x_1 & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}$$

a) Is there a bounded optimal solution? If not, give a radius, along which the unboundedness is provable.

b) If a bounded optimal solution exists, give an optimal solution and the corresponding objective function value.

c) Is the optimum unique? Justify your answer. If it is not, give alternative optimal solutions.

d) How does the optimal objective function value change if we modify the coefficients in the objective function in the following way:

- decrease the coefficient of x_4 to 3,
- decrease the coefficient of x_1 to 1,
- increase the coefficient of x_1 to 9,
- after this last change, increase the coefficient of x_2 to 9?

5. Solve the following linear program using the simplex method:

$$\begin{array}{lrcrcrcr}
\max & x_1 & - & 2x_2 & + & x_3 & & \\
\text{s.t.} & x_1 & + & x_2 & + & x_3 & \leq & 12 \\
 & 2x_1 & + & x_2 & - & x_3 & \leq & 6 \\
 & -x_1 & + & 3x_2 & & & \leq & 9 \\
 & x_1, & & x_2, & & x_3 & \geq & 0
\end{array}$$

a) Is the optimal objective function value of the linear program bounded? If not, give a ray causing the unboundedness!

b) If it exists, provide an optimal solution with the corresponding objective function value.

c) Is the optimal solution unique? If it is, provide a proof. If not, give alternative optimal solutions.

d) How do the optimal solution and the objective function change, if we

- decrease the objective function coefficient for x_1 to -1,
- increase the objective function coefficient for x_1 to 2,

- decrease the objective function coefficient for x_2 to -4,
- increase the objective function coefficient for x_2 to 3?

6. Solve the following linear program using the simplex method:

$$\begin{array}{lrcrcrcrcr}
\text{min} & -2x_1 & - & 2x_2 & + & 3x_3 & - & 5x_4 & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & 4x_3 & - & x_4 & \leq & 6 \\
 & -2x_1 & - & 3x_2 & + & x_3 & - & x_4 & \geq & -12 \\
 & x_1 & & & + & x_3 & + & x_4 & \leq & 4 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}$$

   a) Is the optimal objective function value of the linear program bounded? If not, give a ray causing the unboundedness!
   b) If it exists, provide an optimal solution with the corresponding objective function value.
   c) Is the optimal solution unique? If it is, provide a proof. If not, give alternative optimal solutions.

7. Solve the following linear program using the simplex method:

$$\begin{array}{lrcrcrcrcr}
\text{max} & 3x_1 & + & 2x_2 & - & x_3 & + & x_4 & & \\
\text{s.t.} & 2x_1 & - & 4x_2 & - & x_3 & + & x_4 & \leq & 8 \\
 & x_1 & + & x_2 & + & 2x_3 & - & 3x_4 & \leq & 10 \\
 & x_1 & - & x_2 & - & 4x_3 & + & x_4 & \leq & 4 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}$$

   a) Is the optimal objective function value of the linear program bounded? If not, give a ray causing the unboundedness!
   b) If it exists, provide an optimal solution with the corresponding objective function value.
   c) Is the optimal solution unique? If it is, provide a proof. If not, give alternative optimal solutions.

8. Solve the following linear program using the simplex method:

$$\begin{array}{lrcrcr}
\text{min} & 3x_1 & - & x_2 & & \\
\text{s.t.} & x_1 & - & 3x_2 & \geq & -3 \\
 & 2x_1 & + & 3x_2 & \geq & -6 \\
 & 2x_1 & + & x_2 & \leq & 8 \\
 & 4x_1 & - & x_2 & \leq & 16
\end{array}$$

   a) Is the optimal objective function value of the linear program bounded? If not, give a ray causing the unboundedness!
   b) If it exists, provide an optimal solution with the corresponding objective function value.
   c) Is the optimal solution unique? If it is, provide a proof. If not, give alternative optimal solutions.

9. Solve the following linear program using the simplex method. Take note of the equality- and inequality-types of constraints.

$$\begin{array}{lrcrcrcrcr}
\text{max} & 3x_1 & + & 4x_2 & + & 3x_3 & + & 5x_4 & & \\
\text{s.t.} & 2x_1 & + & x_2 & - & x_3 & + & x_4 & \geq & 11 \\
 & x_1 & + & x_2 & + & x_3 & + & x_4 & = & 8 \\
 & & - & x_2 & + & 2x_3 & + & x_4 & \leq & 10 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}$$

   a) Find an initial basis! Introduce artificial variables, if necessary.
   b) Solve the linear program with corresponding simplex method. Is the optimal objective function value of the linear program bounded? If not, give a ray causing the unboundedness! If the optimal objective function value is bounded, is the corresponding optimal solution unique? If it is, provide a proof. If not, give alternative optimal solutions.

c) Do the optimal solution and the objective function value change, if we
   - decrease the objective function coefficient for x_3 to 1,
   - increase the objective function coefficient for x_3 to 12,
   - decrease the objective function coefficient for x_1 to 1,
   - increase the objective function coefficient for x_1 to 7?

Linear Programming Duality

10. Consider the linear program given in Exercise 4:

$$\begin{array}{lrcrcrcrcr}
\max & 3x_1 & + & 8x_2 & - & 5x_3 & + & 8x_4 & & \\
\text{s.t.} & 2x_1 & + & x_2 & + & x_3 & + & 3x_4 & \leq & 7 \\
 & -x_1 & - & 2x_2 & - & x_3 & - & x_4 & \geq & -2 \\
 & x_1 & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}$$

a) Write the dual of the linear program and convert to standard form! The following table summarizes the rules for obtaining the dual linear program:

	Maximization problem		**Minimization problem**	
Constraint	$\geq$	$\longleftrightarrow$	≤ 0	**Variable**
	$\leq$	$\longleftrightarrow$	≥ 0	
	$=$	$\longleftrightarrow$	arbitrary	
Variable	≥ 0	$\longleftrightarrow$	$\geq$	**Constraint**
	≤ 0	$\longleftrightarrow$	$\leq$	
	arbitrary	$\longleftrightarrow$	$=$	

b) Find an initial basis! Introduce artificial variables, if necessary.

c) Solve the linear program using the corresponding simplex method. Is the optimal objective function value of the linear program bounded? If not, give a ray causing the unboundedness! If the optimal objective function value is bounded, is the corresponding optimal solution unique? If it is, provide a proof. If not, give alternative optimal solutions.

d) Compare the resultant optimal solution with the solution of Exercise 4. What is the relationship between the primal and dual optimal solutions?

Solving Word Problems with the Simplex Algorithm

11. In a paper mill, the machines are being replaced. Two types of cardboard-cutting machines can be purchased: machine A can cut 3 boxes per one minute, one person is needed to operate it, and it costs 15,000 units of money; machine B machine can make 5 boxes per minute, but it requeres two people to supervise it, and it costs 20,000 units of money. The production plan is to produce at least 32 boxes per minute with at most 12 workers involved.

How many A and B machines needs to be purchased to fit the production plan with minimized costs?

a) Define the above "resource acquisition" problem as a linear program!

b) Find an initial basis!

c) Solve the linear program with the primal or the dual simplex algorithm!

d) Got an integer as a result? If so, is integrality of the results guaranteed?

Solutions

Word Problems

1. Four types of hollow structural sections (HSS, a special type of metal bar or tube) are manufactured in a steel mill: small, medium, large and extra large. There are three types of machines available: A, B, and C. The following table specifies the quantity of HSS types (in terms of length, in meters) produced per hour, depending on the type of machine.

	Machine		
Hollow structural section (HSS) type	A	B	C
Small	3	6	8
Medium	2	4	7
Large	2	3	6
Extra large	1	2	3

The machines can operate 50 hours a week and the operational cost of each machine (in units of money) is: 3, 5 and 8. According to the business plan, 200, 80, 60, and 60 meters per week are needed of each of the HSS types A, B, C, and respectively D.

a) Define the above machine scheduling problem as a linear program!

b) Find an initial basis!

c) Would you solve the problem with the primal or the dual simplex algorithm? (No need to solve it!)

Solution:

a) Let x_{ij} mark the number of hours spent by machine i ($i \in \{1,2,3\}$) to produce HSSes of type j ($j \in \{1,2,3,4\}$). Using this notation, the linear program is the following:

$$
\begin{array}{lccccccccccccl}
\min & 3x_{11} & +3x_{12} & +3x_{13} & +3x_{14} & +5x_{21} & +5x_{22} & +5x_{23} & +5x_{24} & +8x_{31} & +8x_{32} & +8x_{33} & +8x_{34} & \\
\text{s.t.} & x_{11} & +x_{12} & +x_{13} & +x_{14} & & & & & & & & & \leq 50 \\
 & & & & & x_{21} & +x_{22} & +x_{23} & +x_{24} & & & & & \leq 50 \\
 & & & & & & & & & +x_{31} & +x_{32} & +x_{33} & +x_{34} & \leq 50 \\
 & 3x_{11} & & & & +6x_{21} & & & & +8x_{31} & & & & = 200 \\
 & & 2x_{12} & & & & +4x_{22} & & & & +7x_{32} & & & = 80 \\
 & & & 2x_{13} & & & & +3x_{23} & & & & +6x_{33} & & = 60 \\
 & & & & x_{14} & & & & +2x_{24} & & & & +3x_{34} & = 60 \\
 & x_{11}, & x_{12}, & x_{13}, & x_{14}, & x_{21}, & x_{22}, & x_{23}, & x_{24}, & x_{31}, & x_{32}, & x_{33}, & x_{34} & \geq 0
\end{array}
$$

b) The fourth, fifth, sixth and seventh condition could be rewritten into $\geq$-form, the optimal objective function value would not change because the goal is to minimize costs. Inverting these conditions to $\leq$-form and rewriting the objective into maximization:

$$
\begin{array}{llllllllllllll}
\max & -3x_{11} & -3x_{12} & -3x_{13} & -3x_{14} & -5x_{21} & -5x_{22} & -5x_{23} & -5x_{24} & -8x_{31} & -8x_{32} & -8x_{33} & -8x_{34} & \\
\text{s.t.} & x_{11} & +x_{12} & +x_{13} & +x_{14} & & & & & & & & & \leq 50 \\
 & & & & & x_{21} & +x_{22} & +x_{23} & +x_{24} & & & & & \leq 50 \\
 & & & & & & & & & +x_{31} & +x_{32} & +x_{33} & +x_{34} & \leq 50 \\
 & -3x_{11} & & & & -6x_{21} & & & & -8x_{31} & & & & \leq -200 \\
 & & -2x_{12} & & & & -4x_{22} & & & & -7x_{32} & & & \leq -80 \\
 & & & -2x_{13} & & & & -3x_{23} & & & & -6x_{33} & & \leq -60 \\
 & & & & -x_{14} & & & & -2x_{24} & & & & -3x_{34} & \leq -60 \\
 & x_{11}, & x_{12}, & x_{13}, & x_{14}, & x_{21}, & x_{22}, & x_{23}, & x_{24}, & x_{31}, & x_{32}, & x_{33}, & x_{34} & \geq 0
\end{array}
$$

We got a canonical problem that should be brought to standard form by introducing slack variables. The coefficients of the objective function are all negative, so the slack variables form a dual-feasible (but not dual-optimal) initial basis.

c) Due to the above, the dual simplex algorithm is recommended.

Solving Linear Programs: The Graphical Method

2. Consider the following linear program (Warning: There are no non-negativity constraints!):

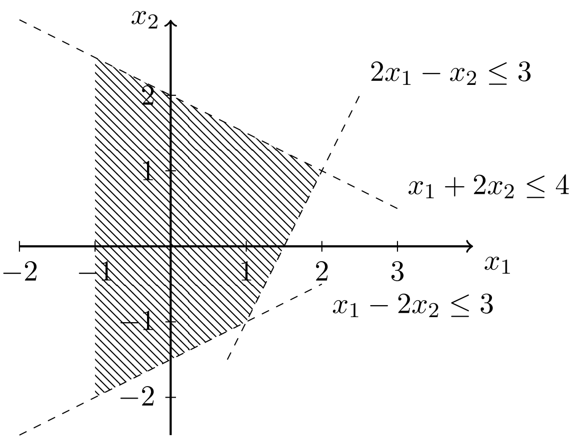
$$
\begin{array}{lrcrcr}
\max & x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & + & 2x_2 & \leq & 4 \\
 & 2x_1 & - & x_2 & \leq & 3 \\
 & x_1 & - & 2x_2 & \leq & 3
\end{array}
$$

a) Solve the linear program *graphically*!

b) Solve the linear program (also graphically), when the objective function is changed to $\max -x_1 + x_2$!

c) Is there a unique optimal solution, if the objective function is $\max\ 2x_1 - x_2$? Verify your answer graphically!

Solution: The feasible region of the given linear program represented graphically:



a) The normal vector of the objective function is the $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ vector, which gives us the following extreme point: $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ as the unique optimal solution.

b) In this case the normal vector of the objective function changes to $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, so the optimal solution of the linear program is unbounded. For example, along the ray $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix}, \lambda \geq 0$ the objective function value grows without limit as $\frac{3}{2}\lambda$.

c) No, the linear program has multiple alternative optimal solutions as follows: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \lambda \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \lambda \in [0, 1]$.

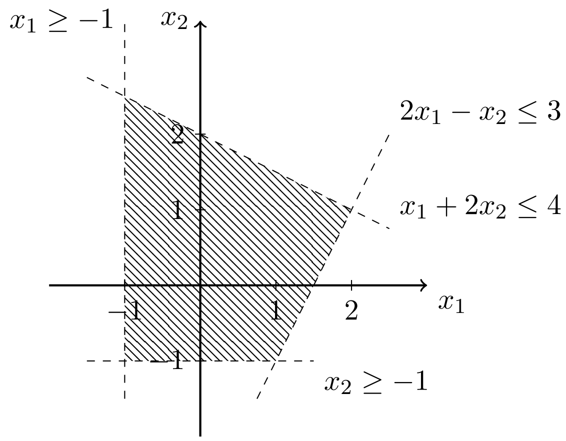
3. Consider the following linear program:

$$\begin{array}{lrcrcr} \min & -x_1 & + & 2x_2 & & \\ \text{s.t.} & x_1 & + & 2x_2 & \leq & 4 \\ & 2x_1 & - & x_2 & \leq & 3 \\ & x_1 & & & \geq & -1 \\ & & & x_2 & \geq & -1 \end{array}$$

a) Illustrate the feasible region graphically!
b) Give the extreme points of the feasible region!
c) Give the optimal value of the objective function and an optimal solution!
d) Is the solution unique? Justify your answer!
e) How do the optimal value of the objective function and the corresponding solution change if we modify the objective function to $\min x_1 + x_2$? Justify your answer graphically as well!

Solution:

a) The graphical representation of the feasible region:



b) The extreme points can be derived easily from the graphical representation:

$$\boldsymbol{x}_1 = \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} -1 \\ 2.5 \end{bmatrix}, \quad \boldsymbol{x}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \boldsymbol{x}_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

c) The feasible region is bounded, thus it is guaranteed that at least one optimal solution will occur at extreme point. The optimal solution is going to be the extreme point where the product $\boldsymbol{c}^T \boldsymbol{x}_e$ takes its minimum value over the set of extreme points $\{\boldsymbol{x}_e\}$:

$$\boldsymbol{c}^T \boldsymbol{x}_1 = \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = -1, \quad \boldsymbol{c}^T \boldsymbol{x}_2 = \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2.5 \end{bmatrix} = 6,$$

$$\boldsymbol{c}^T \boldsymbol{x}_3 = \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 0, \quad \boldsymbol{c}^T \boldsymbol{x}_4 = \begin{bmatrix} -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -3$$

From this the optimal solution is -3, which corresponds to the extreme point $\boldsymbol{x}_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

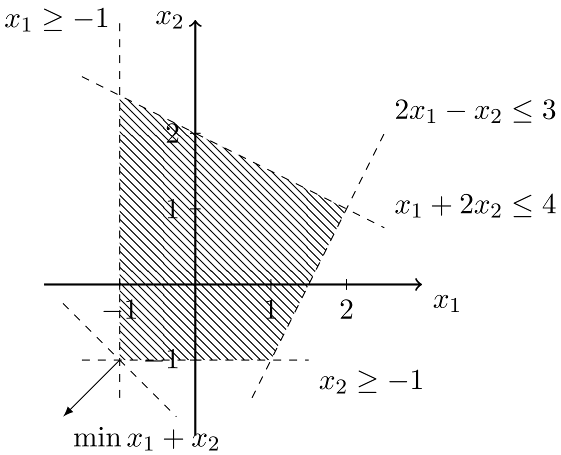
d) The solution is unique because the intersection of the objective function contour $-x_1+2x_2 = -3$ and the feasible region contains a single point only: $\boldsymbol{x}_4 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

e) After changing the objective function from $\boldsymbol{c}^T = \begin{bmatrix}-1 & 2\end{bmatrix}$ to $\boldsymbol{c'}^T = \begin{bmatrix}1 & 1\end{bmatrix}$, the new products $\boldsymbol{c'}^T\boldsymbol{x}_e$ are as follows:

$$\boldsymbol{c'}^T\boldsymbol{x}_1 = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}-1\\-1\end{bmatrix} = -2, \quad \boldsymbol{c'}^T\boldsymbol{x}_2 = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}-1\\2.5\end{bmatrix} = 1.5,$$

$$\boldsymbol{c'}^T\boldsymbol{x}_3 = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix} = 3, \quad \boldsymbol{c}^T\boldsymbol{x}_4 = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = 0$$

From this, the optimal solution is -2, which corresponds to the extreme point $\boldsymbol{x}_1 = \begin{bmatrix}-1\\-1\end{bmatrix}$. Graphically,the optimization of the new objective function means finding the intersection of the feasible region polyhedron and the hyperplane with the normal vector $\begin{bmatrix}-1\\-1\end{bmatrix}$.



Solving Linear Programs: The Simplex Method

4. Solve the following linear program with the simplex method:

$$\begin{array}{lrcrcrcrcr}
\max & 3x_1 & + & 8x_2 & - & 5x_3 & + & 8x_4 & & \\
\text{s.t.} & 2x_1 & + & x_2 & + & x_3 & + & 3x_4 & \leq & 7 \\
 & -x_1 & - & 2x_2 & - & x_3 & - & x_4 & \geq & -2 \\
 & x_1 & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}$$

a) Is there a bounded optimal solution? If not, give a radius, along which the unboundedness is provable.

b) If a bounded optimal solution exists, give an optimal solution and the corresponding objective function value.

c) Is the optimum unique? Justify your answer. If it is not, give alternative optimal solutions.

d) How does the optimal objective function value change if we modify the coefficients in the objective function in the following way:
   - decrease the coefficient of x_4 to 3,
   - decrease the coefficient of x_1 to 1,
   - increase the coefficient of x_1 to 9,
   - after this last change, increase the coefficient of x_2 to 9?

Solution: After inverting the second condition and introducing the slack variables x_5 and x_6, the a slack variables form a trivial initial basis. Because of this, we use the primal simplex algorithm. The initial simplex tableau:

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	-3	-8	5	-8	0	0	0
x_5	0	2	1	1	3	1	0	7
x_6	0	1	$\boxed{2}$	1	1	0	1	2

x_2 enters the basis, x_6 leaves.

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	1	0	9	-4	0	4	8
x_5	0	$\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{5}{2}$	1	$-\frac{1}{2}$	6
x_2	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$\boxed{\frac{1}{2}}$	0	$\frac{1}{2}$	1

x_4 enters the basis, x_2 leaves.

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	5	8	13	0	0	8	16
x_5	0	-1	-5	-2	0	1	-3	1
x_4	0	1	2	1	1	0	1	2

Optimal simplex tableau. The solution is $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$, the objective function value is 16. The solution thus exists, and it is bounded and unique.

To solve *d)*, we have to perform sensitivity analysis on the optimal simplex tableau with the new objective function.

- the coefficient of x_4 in the objective function is decreased to 3: x_4 is a basic variable in the optimal simplex tableau, so in order to perform sensitivity analysis we have to modify the simplex tableau in the following way. We have to multiply the row of x_4 by $c_4' - c_4 = 3 - 8 = -5$ and add that to the 0th row. In addition, we need to make sure that the reduced cost corresponding to x_4 is zero.

  The resultant tableau is not primal optimal:

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	-2	8	0	0	3	6
x_5	0	-1	-5	-2	0	1	-3	1
x_4	0	1	$\boxed{2}$	1	1	0	1	2

  x_4 leaves the basis, x_2 enters; the resultant tableau is optimal:

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	1	0	9	1	0	4	8
x_5	0	$\frac{3}{2}$	0	$\frac{1}{2}$	$\frac{5}{2}$	1	$-\frac{1}{2}$	6
x_2	0	$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	1

The result of the sensitivity analysis: if the coefficient of x_2 in the objective function is decreased to 3, the optimal solution is $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ (bounded and unique), and the new objective function value is 8.

- the coefficient of x_1 in the objective function decreases to 1: x_1 is a nonbasic variable in the optimal simplex tableau, thus changing its coefficient results in increased reduced cost ($z_1' = z_1 - (c_1' - c_1) = 5 - (1 - 3) = 7 \geq 0$), and the tableau is guaranteed to remain optimal. This means that the solution does not change.
- the coefficient of x_1 in the objective function increases to 9: in this case the reduced cost of the nonbasic variable x_1 decreases: $z_1' = z_1 - (c_1' - c_1) = 5 - (9 - 3) = -1$. Since z_1' has become negative, the tableau is no longer optimal and we have to optimize it again using the primal simplex algorithm.

  The new simplex tableau:

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	-1	8	13	0	0	8	16
x_5	0	-1	-5	-2	0	1	-3	1
x_4	0	$\boxed{1}$	2	1	1	0	1	2

  x_1 enters the basis, x_4 leaves, the tableau after the pivot is optimal:

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	10	14	1	0	9	18
x_5	0	0	-3	-1	1	1	-2	3
x_1	0	1	2	1	1	0	1	2

  The result of the sensitivity analysis: increasing the coefficient of x_1 to 9 results in the optimal solution $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ (bounded and unique), and the new objective function value is 18.
- After this last change, the coefficient of x_2 increases to 9: x_2 is a nonbasic variable in the resultant optimal tableau, its cost decreases after the change: $z_2' = z_2 - (c_2' - c_2) = 10 - (9 - 8) = 9 > 0$, thus the tableau remains optimal.

5. Solve the following linear program using the simplex method:

$$\begin{array}{llcrcrcrcl} \max & x_1 & - & 2x_2 & + & x_3 & & \\ \text{s.t.} & x_1 & + & x_2 & + & x_3 & \leq & 12 \\ & 2x_1 & + & x_2 & - & x_3 & \leq & 6 \\ & -x_1 & + & 3x_2 & & & \leq & 9 \\ & x_1, & & x_2, & & x_3 & \geq & 0 \end{array}$$

   a) Is the optimal objective function value of the linear program bounded? If not, give a ray causing the unboundedness!
   b) If it exists, provide an optimal solution with the corresponding objective function value.
   c) Is the optimal solution unique? If it is, provide a proof. If not, give alternative optimal solutions.
   d) How do the optimal solution and the objective function change, if we
      - decrease the objective function coefficient for x_1 to -1,

- increase the objective function coefficient for x_1 to 2,
- decrease the objective function coefficient for x_2 to -4,
- increase the objective function coefficient for x_2 to 3?

Solution: Convert the constraint system to standard form, introduce the slack variables, and use the primal simplex method. Choose the identity matrix introduced by the slack variables as the initial basis. After the simplex iteration steps, the optimal simplex tableau is the following:

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	3	0	1	0	0	12
x_3	0	0	$\frac{1}{3}$	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	6
x_1	0	1	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	0	6
x_6	0	0	$\frac{11}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	1	15

This tableau gives us the optimal basic feasible solution: $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{bmatrix}^T = \begin{bmatrix} 6 & 0 & 6 & 0 & 0 & 15 \end{bmatrix}^T$. The optimal solution in the original variable space: $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T = \begin{bmatrix} 6 & 0 & 6 \end{bmatrix}^T$, and the objective function value is 12.

The non-basic variable x_5 get the reduced cost coefficient $z_5 = 0$ in row 0 and the basis is non-degenerate, therefore the optimal solution is not unique. For example, one definition for the alternative optimal solutions:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6 \\ 0 \\ 0 \\ 15 \end{bmatrix} - \lambda \begin{bmatrix} \frac{1}{3} \\ 0 \\ -\frac{1}{3} \\ 0 \\ -1 \\ \frac{1}{3} \end{bmatrix}, \qquad 0 \le \lambda \le 18$$

The same ray in the space of the original variables:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6 \end{bmatrix} + \lambda \begin{bmatrix} -\frac{1}{3} \\ 0 \\ \frac{1}{3} \end{bmatrix}, \qquad 0 \le \lambda \le 18$$

To answer *d)* we have to perform sensitivity analysis on the optimal simplex tableau by changing the objective function.

- The objective coefficient of x_1 is decreased to -1: In the optimal simplex tableau x_1 is a basic variable, so to perform the analysis we have to add the row that belongs to x_1 exactly $c'_1 - c_1 = -1 - 1 = -2$ times to row 0 and set the reduced cost for x_1 to zero.
  The resultant tableau is not optimal:

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	$\frac{5}{3}$	0	$\frac{1}{3}$	$-\frac{2}{3}$	0	0
x_3	0	0	$\frac{1}{3}$	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	6
x_1	0	1	$\frac{2}{3}$	0	$\frac{1}{3}$	$\boxed{\frac{1}{3}}$	0	6
x_6	0	0	$\frac{11}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	1	15

Using the primal simplex, x_5 enters the basis and x_1 leaves it.

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	2	3	0	1	0	0	12
x_3	0	1	1	1	1	0	0	12
x_5	0	3	2	0	1	1	0	18
x_6	0	-1	3	0	0	0	1	9

The new tableau is optimal. The result of the sensitivity analysis: Reducing the objective function coefficient of x_1 to -1, the optimal solution becomes $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$, (bounded and unique), and the objective function value remains 12. Observe that we'd get the same x point as an alternative optimal solution in the original linear program for the choice $\lambda = 18$.

- The objective coefficient of x_1 is increased to 2: In this case we add the row corresponding to x_1 $c_1' - c_1 = 2 - 1 = 1$ times to row 0.

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	$\frac{11}{3}$	0	$\frac{4}{3}$	$\frac{1}{3}$	0	18
x_3	0	0	$\frac{1}{3}$	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	6
x_1	0	1	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	0	6
x_6	0	0	$\frac{11}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	1	15

The resultant tableau is optimal. The solution does not change, but the objective function value does: doubling the "value" of x_1 increases the optimum to 18. The solution becomes unique, in contrast to the original linear programban.

- The objective coefficient of x_2 is decreased to -4: Since changing a non-basic objective function coefficient never changes the optimal solution, the tableau remains optimal.
- The objective coefficient of x_2 is increased to 3: In row 0 the reduced cost value for x_2 becomes negative, because $z_2' = z_2 - (c_2' - c_2) = 3 - (3 - (-2)) = -2 < 0$. Therefore the solution is no longer optimal. The changed tableau:

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	-2	0	1	0	0	12
x_3	0	0	$\frac{1}{3}$	1	$\frac{2}{3}$	$-\frac{1}{3}$	0	6
x_1	0	1	$\frac{2}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	0	6
x_6	0	0	$\frac{11}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	1	15

The optimal tableau:

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	0	0	$\frac{13}{11}$	$\frac{2}{11}$	$\frac{6}{11}$	$\frac{222}{11}$
x_3	0	0	0	1	$\frac{7}{11}$	$-\frac{4}{11}$	$-\frac{1}{11}$	$\frac{51}{11}$
x_1	0	1	0	0	$\frac{3}{11}$	$\frac{3}{11}$	$-\frac{2}{11}$	$\frac{36}{11}$
x_2	0	0	1	0	$\frac{1}{11}$	$\frac{1}{11}$	$\frac{3}{11}$	$\frac{45}{11}$

The result of the sensitivity analysis: Reducing the objective function coefficient of x_1 to -1, the optimal solution becomes $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{36}{11} \\ \frac{45}{11} \\ \frac{51}{11} \end{bmatrix}$, (bounded and unique) and the new objective function value is $\frac{222}{11}$.

6. Solve the following linear program using the simplex method:

$$\begin{array}{lrcrcrcrcr} \min & -2x_1 & - & 2x_2 & + & 3x_3 & - & 5x_4 & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & 4x_3 & - & x_4 & \leq & 6 \\ & -2x_1 & - & 3x_2 & + & x_3 & - & x_4 & \geq & -12 \\ & x_1 & & & + & x_3 & + & x_4 & \leq & 4 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

a) Is the optimal objective function value of the linear program bounded? If not, give a ray causing the unboundedness!

b) If it exists, provide an optimal solution with the corresponding objective function value.

c) Is the optimal solution unique? If it is, provide a proof. If not, give alternative optimal solutions.

Solution: Convert the linear program to maximization form by inverting the objective function. Invert the second constraint as well to get a primal feasible identity matrix formed by the slack variables. Using the primal simplex algorithm, the optimal simplex tableau is the following:

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	1	$\frac{11}{3}$	0	$\frac{20}{3}$	0	0	$\frac{2}{3}$	$\frac{13}{3}$	$\frac{76}{3}$
x_5	0	$\frac{4}{3}$	0	$\frac{19}{3}$	0	1	$-\frac{2}{3}$	$\frac{5}{3}$	$\frac{14}{3}$
x_2	0	$\frac{1}{3}$	1	$-\frac{2}{3}$	0	0	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{8}{3}$
x_4	0	1	0	1	1	0	0	1	4

Because the original objective function was a minimization, we have to invert the result. Therefore, the optimal solution arises at $\begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T = \begin{bmatrix} 0 & -\frac{8}{3} & 0 & -4 \end{bmatrix}^T$ where the objective function value is $-\frac{76}{3}$. The optimal solution is unique.

7. Solve the following linear program using the simplex method:

$$\begin{array}{lrcrcrcrcr} \max & 3x_1 & + & 2x_2 & - & x_3 & + & x_4 & & \\ \text{s.t.} & 2x_1 & - & 4x_2 & - & x_3 & + & x_4 & \leq & 8 \\ & x_1 & + & x_2 & + & 2x_3 & - & 3x_4 & \leq & 10 \\ & x_1 & - & x_2 & - & 4x_3 & + & x_4 & \leq & 4 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

a) Is the optimal objective function value of the linear program bounded? If not, give a ray causing the unboundedness!

b) If it exists, provide an optimal solution with the corresponding objective function value.

c) Is the optimal solution unique? If it is, provide a proof. If not, give alternative optimal solutions.

Solution: The optimal solution of the linear program is unbounded. The initial simplex tableau:

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	1	-3	-2	1	-1	0	0	0	0
x_5	0	2	-4	-1	1	1	0	0	8
x_6	0	1	1	2	-3	0	1	0	10
x_7	0	$\boxed{1}$	-1	-4	1	0	0	1	4

Variable x_1 enters, and variable x_7 leaves the basis (x_5 could also be the leaving variable).

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	1	0	-5	-11	2	0	0	3	12
x_5	0	0	-2	$\boxed{7}$	-1	1	0	-2	0
x_6	0	0	2	6	-4	0	1	-1	6
x_1	0	1	-1	-4	1	0	0	1	4

After a degenerate pivot:

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	1	0	$-\frac{57}{7}$	0	$\frac{3}{7}$	$\frac{11}{7}$	0	$-\frac{1}{7}$	12
x_3	0	0	$-\frac{2}{7}$	1	$-\frac{1}{7}$	$\frac{1}{7}$	0	$-\frac{2}{7}$	0
x_6	0	0	$\boxed{\frac{26}{7}}$	0	$-\frac{22}{7}$	$-\frac{6}{7}$	1	$\frac{5}{7}$	6
x_1	0	1	$-\frac{15}{7}$	0	$\frac{3}{7}$	$\frac{4}{7}$	0	$-\frac{1}{7}$	4

The final tableau of the primal simplex:

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	1	0	0	0	$-\frac{84}{13}$	$-\frac{4}{13}$	$\frac{57}{26}$	$\frac{37}{26}$	$\frac{327}{13}$
x_3	0	0	0	1	$-\frac{5}{13}$	$\frac{1}{13}$	$\frac{1}{13}$	$-\frac{3}{13}$	$\frac{6}{13}$
x_2	0	0	1	0	$-\frac{11}{13}$	$-\frac{3}{13}$	$\frac{7}{26}$	$\frac{5}{26}$	$\frac{21}{13}$
x_1	0	1	0	0	$-\frac{18}{13}$	$\frac{1}{13}$	$\frac{15}{26}$	$\frac{7}{26}$	$\frac{97}{13}$

The tableau shows that x_4 can be increased without limit. The ray causing the unboundedness in the space of the original variables (without the slack variables):

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} \frac{97}{13} \\ \frac{21}{13} \\ \frac{6}{13} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} \frac{18}{13} \\ \frac{11}{13} \\ \frac{5}{13} \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \lambda \geq 0 \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{97}{213} \\ \frac{21}{13} \\ \frac{6}{13} \\ 0 \end{bmatrix} + \lambda \begin{bmatrix} \frac{18}{13} \\ \frac{11}{13} \\ \frac{5}{13} \\ 1 \end{bmatrix}, \lambda \geq 0$$

Meanwhile, the objective function grows according to $\frac{327}{13} + \lambda \frac{84}{13}, \lambda \geq 0$.

8. Solve the following linear program using the simplex method:

$$\begin{array}{lrcrcr} \min & 3x_1 & - & x_2 & & \\ \text{s.t.} & x_1 & - & 3x_2 & \geq & -3 \\ & 2x_1 & + & 3x_2 & \geq & -6 \\ & 2x_1 & + & x_2 & \leq & 8 \\ & 4x_1 & - & x_2 & \leq & 16 \end{array}$$

a) Is the optimal objective function value of the linear program bounded? If not, give a ray causing the unboundedness!

b) If it exists, provide an optimal solution with the corresponding objective function value.

c) Is the optimal solution unique? If it is, provide a proof. If not, give alternative optimal solutions.

Solution: Observe that there is no non-negativity/non-positivity constraints! Substituting $x_1 = y_1 - y_2, y_1 \geq 0, y_2 \geq 0$ and $x_2 = y_3 - y_4, y_3 \geq 0, y_4 \geq 0$, the maximization problem in canonical form is the following:

$$\begin{array}{lrcrcrcrcr}
\max & -3y_1 & + & 3y_2 & + & y_3 & - & y_4 & & \\
\text{s.t.} & -y_1 & + & y_2 & + & 3y_3 & - & 3y_4 & \leq & 3 \\
& -2y_1 & + & 2y_2 & - & 3y_3 & + & 3y_4 & \leq & 6 \\
& 2y_1 & - & 2y_2 & + & y_3 & - & y_4 & \leq & 8 \\
& 4y_1 & - & 4y_2 & - & y_3 & + & y_4 & \leq & 16 \\
& y_1, & & y_2, & & y_3, & & y_4 & \geq & 0
\end{array}$$

Introduce slack variables to bring the problem into standard form. After executing the primal simplex, the optimal simplex tableau is the following:

	z	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	RHS
z	1	0	0	0	0	$\frac{11}{9}$	$\frac{8}{9}$	0	0	9
y_2	0	-1	1	0	0	$\frac{1}{3}$	$\frac{1}{3}$	0	0	3
y_4	0	0	0	-1	1	$-\frac{2}{9}$	$\frac{1}{9}$	0	0	0
y_7	0	0	0	0	0	$\frac{4}{9}$	$\frac{7}{9}$	1	0	14
y_8	0	0	0	0	0	$\frac{14}{9}$	$\frac{11}{9}$	0	1	28

The optimum of the original minimization problem is -9, which arises at the point $x_1 = y_1 - y_2 = -3$, $x_2 = y_3 - y_4 = 0$. Observe in the optimal simplex tableau that in the column corresponding to y_1 (which is the first column) the value corresponding to y_2 is -1. Therefore, increasing y_1 would increase the value of y_2 by the same quantity and $x_1 = y_1 - y_2$ would remain the same. This is also true for the variables y_3 and y_4. Accordingly, such substitutions for free variables usually yield an optimal tableau with alternative optimal solutions. In this case, we see that the linear program has infinite number of solutions but all solutions effectively belong to the same point $\begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$ in the original linear program.

9. Solve the following linear program using the simplex method. Take note of the equality- and inequality-types of constraints.

$$\begin{array}{lrcrcrcrcr}
\max & 3x_1 & + & 4x_2 & + & 3x_3 & + & 5x_4 & & \\
\text{s.t.} & 2x_1 & + & x_2 & - & x_3 & + & x_4 & \geq & 11 \\
& x_1 & + & x_2 & + & x_3 & + & x_4 & = & 8 \\
& & - & x_2 & + & 2x_3 & + & x_4 & \leq & 10 \\
& x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}$$

a) Find an initial basis! Introduce artificial variables, if necessary.

b) Solve the linear program with corresponding simplex method. Is the optimal objective function value of the linear program bounded? If not, give a ray causing the unboundedness! If the optimal objective function value is bounded, is the corresponding optimal solution unique? If it is, provide a proof. If not, give alternative optimal solutions.

c) Do the optimal solution and the objective function value change, if we

- decrease the objective function coefficient for x_3 to 1,
- increase the objective function coefficient for x_3 to 12,
- decrease the objective function coefficient for x_1 to 1,
- increase the objective function coefficient for x_1 to 7?

Solution:

a) Transform to standard form with introducing slack-variables:

$$\begin{array}{lrcrcrcrcrcrcl}
\max & 3x_1 & + & 4x_2 & + & 3x_3 & + & 5x_4 & & & & & & \\
\text{s.t.} & 2x_1 & + & x_2 & - & x_3 & + & x_4 & - & x_5 & & & = & 11 \\
& x_1 & + & x_2 & + & x_3 & + & x_4 & & & & & = & 8 \\
& & - & x_2 & + & 2x_3 & + & x_4 & & & + & x_6 & = & 10 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0
\end{array}$$

There is no trivial initial unit basis, but we can use the slack-variable for the third constraint (x_6) as one candidate basic variable. To obtain the remaining two basic variables, we introduce the artificial variables x_7 and x_8.

$$\begin{array}{lrcrcrcrcrcrcrcrcl}
\max & 3x_1 & + & 4x_2 & + & 3x_3 & + & 5x_4 & & & & & & & & & & \\
\text{s.t.} & 2x_1 & + & x_2 & - & x_3 & + & x_4 & - & x_5 & & & + & x_7 & & & = & 11 \\
& x_1 & + & x_2 & + & x_3 & + & x_4 & & & & & & & + & x_8 & = & 8 \\
& & - & x_2 & + & 2x_3 & + & x_4 & & & + & x_6 & & & & & = & 10 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & & x_7, & & x_8 & \geq & 0
\end{array}$$

So, in the first phase the primal simplex method is used for the initial unit basis formed by the columns of x_6, x_7 and x_8. The objective is to remove the artificial variables x_7 and x_8 from the basis.

$$\begin{array}{lrcrcrcrcrcrcrcrcl}
\min & & & & & & & & & & & & & x_7 & + & x_8 & & \\
\text{s.t.} & 2x_1 & + & x_2 & - & x_3 & + & x_4 & - & x_5 & & & + & x_7 & & & = & 11 \\
& x_1 & + & x_2 & + & x_3 & + & x_4 & & & & & & & + & x_8 & = & 8 \\
& & - & x_2 & + & 2x_3 & + & x_4 & & & + & x_6 & & & & & = & 10 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & & x_7, & & x_8 & \geq & 0
\end{array}$$

Change the direction of the optimization to maximization (do not forget to multiply with -1 at the end). The tableau of the first phase in the (unit) basis given by $B = \{x_7, x_8, x_6\}$ is as follows:

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
z	1	0	0	0	0	0	0	$\boxed{1}$	$\boxed{1}$	0
x_7	0	2	1	-1	1	-1	0	1	0	11
x_8	0	1	1	1	1	0	0	0	1	8
x_6	0	0	-1	2	1	0	1	0	0	10

At this point this is still not a valid simplex tableau; for this the framed reduced cost coefficients must be reset in row zero using elementary row operations (adding/subtracting rows). The resultant tableau is as follows:

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
z	1	-3	-2	0	-2	1	0	0	0	-19
x_7	0	$\boxed{2}$	1	-1	1	-1	0	1	0	11
x_8	0	1	1	1	1	0	0	0	1	8
x_6	0	0	-1	2	1	0	1	0	0	10

b) Apply the primal simplex method for solving the first phase. At the first pivot iteration, x_1 enters basis and x_7 leaves, etc. The optimal tableau for the first phase of the simplex is as follows:

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	RHS
z	1	0	0	0	0	0	0	1	1	0
x_1	0	1	$\frac{2}{3}$	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{19}{3}$
x_3	0	0	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	0	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{5}{3}$
x_6	0	0	$-\frac{5}{3}$	0	$\frac{1}{3}$	$-\frac{2}{3}$	1	$\frac{2}{3}$	$-\frac{4}{3}$	$\frac{20}{3}$

The optimal objective function value for the first phase is 0, so the original problem is feasible. The artificial have left the basis, thus the columns of x_7 and x_8 can be removed and columns for x_1, x_3 and x_6 are available as an initial basis in the second phase.
Restoring the original objective function (do not forget invert the coefficients):

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	$\boxed{-3}$	-4	$\boxed{-3}$	-5	0	0	0
x_1	0	1	$\frac{2}{3}$	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{19}{3}$
x_3	0	0	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{5}{3}$
x_6	0	0	$-\frac{5}{3}$	0	$\frac{1}{3}$	$-\frac{2}{3}$	1	$\frac{20}{3}$

Reset the framed reduced coefficient to get a valid simplex tableau:

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	-1	0	-2	0	0	24
x_1	0	1	$\frac{2}{3}$	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{19}{3}$
x_3	0	0	$\frac{1}{3}$	1	$\boxed{\frac{1}{3}}$	$\frac{1}{3}$	0	$\frac{5}{3}$
x_6	0	0	$-\frac{5}{3}$	0	$\frac{1}{3}$	$-\frac{2}{3}$	1	$\frac{20}{3}$

The second phase starts, x_4 enters the basis and x_3 leaves.

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	1	6	0	2	0	34
x_1	0	1	0	-2	0	-1	0	3
x_4	0	0	1	3	1	1	0	5
x_6	0	0	-2	-1	0	-1	1	5

The resultant tableau is optimal. The optimal objective function value is 34 (bounded) and the solution is unique, because the reduced cost coefficients for all nonbasic variables are strictly positive in row zero.

c) We have to perform sensitivity analysis on the optimal simplex tableau by changing the objective function.

- the objective function coefficient of x_3 is decreased to 1: decreasing of the objective coefficient of any nonbasic variable does not modify the solution, so the tableau remains optimal.
- the objective function coefficient of x_3 is increased to 12: the reduced cost of x_3 becomes negative in row zero, because $z_3' = z_3 - (c_3' - c_3) = 6 - (12 - 3) = -3 < 0$. Therefore, the solution is no longer optimal. The modified tableau is:

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	1	-3	0	2	0	34
x_1	0	1	0	-2	0	-1	0	3
x_4	0	0	1	$\boxed{3}$	1	1	0	5
x_6	0	0	-2	-1	0	-1	1	5

x_3 enters and x_4 leaves the basis, and the resultant simplex tableau is optimal:

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	2	0	1	3	0	39
x_1	0	1	$\frac{2}{3}$	0	$\frac{2}{3}$	$-\frac{1}{3}$	0	$\frac{19}{3}$
x_3	0	0	$\frac{1}{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$	0	$\frac{5}{3}$
x_6	0	0	$-\frac{5}{3}$	0	$\frac{1}{3}$	$-\frac{2}{3}$	1	$\frac{20}{3}$

The result of the sensitivity analysis: by increasing the objective function coefficient of x_3 to -1, x_3 and x_4 change place in the optimal basis and so the new optimal solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{19}{3} \\ 0 \\ \frac{5}{3} \\ 0 \end{bmatrix}$. The objective function value increases to 39.

- the objective function coefficient of x_1 is decreased to 1: x_1 is a basic variable in the optimal solution, so we need to modify the tableau as follows: row of x_1 is multiplied with $c_1' - c_1 = 1 - 3 = -2$ and added to row zero, and the coefficient for x_1 is reset in row zero.

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	1	10	0	4	0	28
x_1	0	1	0	-2	0	-1	0	3
x_4	0	0	1	3	1	1	0	5
x_6	0	0	-2	-1	0	-1	1	5

The tableau remains optimal, however the optimal objective function value decreases to 28.

- the objective function coefficient of x_1 is increased to 7: now, $c_1' - c_1 = 7 - 3 = 4$ times the row of x_1 is added to row zero.

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	1	-2	0	-2	0	46
x_1	0	1	0	-2	0	-1	0	3
x_4	0	0	1	$\boxed{3}$	1	1	0	5
x_6	0	0	-2	-1	0	-1	1	5

x_3 enters the basis and x_4 leaves. The final optimal simplex tableau is:

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	3	4	2	0	0	56
x_1	0	1	1	1	1	0	0	8
x_5	0	0	1	3	1	1	0	5
x_6	0	0	-1	2	1	0	1	10

The result of the sensitivity analysis: the new optimal solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ and the optimal objective function value increases to 56.

Linear Programming Duality

10. Consider the linear program given in Exercise 4:

$$
\begin{array}{lrcrcrcrcr}
\max & 3x_1 & + & 8x_2 & - & 5x_3 & + & 8x_4 & & \\
\text{s.t.} & 2x_1 & + & x_2 & + & x_3 & + & 3x_4 & \leq & 7 \\
 & -x_1 & - & 2x_2 & - & x_3 & - & x_4 & \geq & -2 \\
 & x_1 & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}
$$

a) Write the dual of the linear program and convert to standard form! The following table summarizes the rules for obtaining the dual linear program:

	Maximization problem		**Minimization problem**	
Constraint	$\geq$	$\longleftrightarrow$	≤ 0	**Variable**
	$\leq$	$\longleftrightarrow$	≥ 0	
	$=$	$\longleftrightarrow$	arbitrary	
Variable	≥ 0	$\longleftrightarrow$	$\geq$	**Constraint**
	≤ 0	$\longleftrightarrow$	$\leq$	
	arbitrary	$\longleftrightarrow$	$=$	

b) Find an initial basis! Introduce artificial variables, if necessary.

c) Solve the linear program using the corresponding simplex method. Is the optimal objective function value of the linear program bounded? If not, give a ray causing the unboundedness! If the optimal objective function value is bounded, is the corresponding optimal solution unique? If it is, provide a proof. If not, give alternative optimal solutions.

d) Compare the resultant optimal solution with the solution of Exercise 4. What is the relationship between the primal and dual optimal solutions?

Solution:

a) Using dual variables w_1 and w_2', the dual linear program is as follows:

$$
\begin{array}{lrcrcr}
\min & 7w_1 & - & 2w_2' & & \\
\text{s.t.} & 2w_1 & - & w_2' & \geq & 3 \\
 & w_1 & - & 2w_2' & \geq & 8 \\
 & w_1 & - & w_2' & \geq & -5 \\
 & 3w_1 & - & w_2' & \geq & 8 \\
 & w_1 & & & \geq & 0 \\
 & & & w_2' & \leq & 0
\end{array}
$$

To obtain the standard form, we need to perform the following changes:

- we must change the optimization direction to maximization with inverting the objective function (the resultant solution will need to be multiplied by -1),
- now w_2' is nonpositive, so it must be converted to nonnegative variable using the substitution $w_2 = -w_2'$ (the sign of the coefficients also change),
- finally, slack-variables must be introduced for all rows.

The given standard form:

$$
\begin{array}{lrcrcrcrcrcrcr}
\max & -7w_1 & - & 2w_2 & & & & & & & & & & \\
\text{s.t.} & 2w_1 & + & w_2 & - & w_3 & & & & & & & = & 3 \\
 & w_1 & + & 2w_2 & & & - & w_4 & & & & & = & 8 \\
 & w_1 & + & w_2 & & & & & - & w_5 & & & = & -5 \\
 & 3w_1 & + & w_2 & & & & & & & - & w_6 & = & 8 \\
 & w_1, & & w_2, & & w_3 & & w_4 & & w_5 & & w_6 & \geq & 0
\end{array}
$$

b) The slack-variables give initial dual-feasible (primal-optimal) basis.

c) Due to the above we could use the dual simplex method right away. Still, for the sake of the exercise, let us introduce artificial variables so that we can use the primal simplex.
In the first step, the coefficients in the RHS column must be made positive. Thus, invert the third condition to obtain:

$$
\begin{array}{lrcrcrcrcrcrcr}
\max & -7w_1 & - & 2w_2 & & & & & & & & & & \\
\text{s.t.} & 2w_1 & + & w_2 & - & w_3 & & & & & & & = & 3 \\
 & w_1 & + & 2w_2 & & & - & w_4 & & & & & = & 8 \\
 & -w_1 & - & w_2 & & & & & + & w_5 & & & = & 5 \\
 & 3w_1 & + & w_2 & & & & & & & - & w_6 & = & 8 \\
 & w_1, & & w_2, & & w_3 & & w_4 & & w_5 & & w_6 & \geq & 0
\end{array}
$$

The slack-variable w_5 can be used as one basic variable, for the rest introduce the artificial variables w_7, w_8 and w_9. The objective function is modified to eliminate the artificial variables: $\min w_7 + w_8 + w_9 = -\max -w_7 - w_8 - w_9$. (Do not forget to invert the objective function values at the end!):

$$
\begin{array}{lrcrcrcrcrcrcrcrcrcr}
\max & & & & & & & & & & & & & -1 & - & 1 & - & 1 & & \\
\text{s.t.} & 2w_1 & + & w_2 & - & w_3 & & & & & & & + & w_7 & & & & & = & 3 \\
 & w_1 & + & 2w_2 & & & - & w_4 & & & & & & & + & w_8 & & & = & 8 \\
 & -w_1 & - & w_2 & & & & & + & w_5 & & & & & & & & & = & 5 \\
 & 3w_1 & + & w_2 & & & & & & & - & w_6 & & & & & + & w_9 & = & 8 \\
 & w_1, & & w_2, & & w_3, & & w_4, & & w_5, & & w_6, & & w_7, & & w_8, & & w_9 & \geq & 0
\end{array}
$$

The initial basis is the identity matrix for the columns of variables w_5, w_7, w_8 and w_9.

d) In tabular form (do not forget to invert row zero):

	z	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	RHS
z	1	0	0	0	0	0	0	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	0
w_7	0	2	1	−1	0	0	0	1	0	0	3
w_8	0	1	2	0	−1	0	0	0	1	0	8
w_5	0	−1	−1	0	0	1	0	0	0	0	5
w_9	0	3	1	0	0	0	−1	0	0	1	8

The resultant form is still not a simplex tableau, because the framed elements in row zero are non-zero. Subtract rows of w_7, w_8 and w_9 from row zero to obtain a valid tableau:

	z	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	RHS
z	1	−6	−4	1	1	0	1	0	0	0	−19
w_7	0	$\boxed{2}$	1	−1	0	0	0	1	0	0	3
w_8	0	1	2	0	−1	0	0	0	1	0	8
w_5	0	−1	−1	0	0	1	0	0	0	0	5
w_9	0	3	1	0	0	0	−1	0	0	1	8

First phase starts, w_1 enters the basis and w_7 leaves.

	z	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	RHS
z	1	0	-1	-2	1	0	1	3	0	0	-10
w_1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	$\frac{1}{2}$	0	0	$\frac{3}{2}$
w_8	0	0	$\frac{3}{2}$	$\frac{1}{2}$	-1	0	0	$-\frac{1}{2}$	1	0	$\frac{13}{2}$
w_5	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	1	0	$\frac{1}{2}$	0	0	$\frac{13}{2}$
w_9	0	0	$-\frac{1}{2}$	$\boxed{\frac{3}{2}}$	0	0	-1	$-\frac{3}{2}$	0	1	$\frac{7}{2}$

w_3 enters and w_9 leaves the basis.

	z	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	RHS
z	1	0	$-\frac{5}{3}$	0	1	0	$-\frac{1}{3}$	1	0	$\frac{4}{3}$	$-\frac{16}{3}$
w_1	0	1	$\frac{1}{3}$	0	0	0	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{8}{3}$
w_8	0	0	$\frac{5}{3}$	0	-1	0	$\frac{1}{3}$	0	1	$-\frac{1}{3}$	$\frac{16}{3}$
w_5	0	0	$-\frac{2}{3}$	0	0	1	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	$\frac{23}{3}$
w_3	0	0	$-\frac{1}{3}$	1	0	0	$-\frac{2}{3}$	-1	0	$\frac{2}{3}$	$\frac{7}{3}$

w_2 enters the basis and w_8 leaves.

	z	w_1	w_2	w_3	w_4	w_5	w_6	w_7	w_8	w_9	RHS
z	1	0	0	0	0	0	0	1	1	1	0
w_1	0	1	0	0	$\frac{1}{5}$	0	$-\frac{2}{5}$	0	$-\frac{1}{5}$	$\frac{2}{5}$	$\frac{8}{5}$
w_2	0	0	1	0	$-\frac{3}{5}$	0	$\frac{1}{5}$	0	$\frac{3}{5}$	$-\frac{1}{5}$	$\frac{16}{5}$
w_5	0	0	0	0	$-\frac{2}{5}$	1	$-\frac{1}{5}$	0	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{49}{5}$
w_3	0	0	0	1	$-\frac{1}{5}$	0	$-\frac{3}{5}$	-1	$\frac{1}{5}$	$\frac{3}{5}$	$\frac{17}{5}$

At this point we get the optimal tableau for phase 1. The objective function value is 0, so the original problem is feasible, and the artificial variables have left the basis so columns for w_7, w_8 and w_9 can be removed.
Restore the objective function to the objective of the dual (do not forget invert the coefficients):

	z	w_1	w_2	w_3	w_4	w_5	w_6	RHS
z	1	$\boxed{7}$	$\boxed{2}$	0	0	0	0	0
w_1	0	1	0	0	$\frac{1}{5}$	0	$-\frac{2}{5}$	$\frac{8}{5}$
w_2	0	0	1	0	$-\frac{3}{5}$	0	$\frac{1}{5}$	$\frac{16}{5}$
w_5	0	0	0	0	$-\frac{2}{5}$	1	$-\frac{1}{5}$	$\frac{49}{5}$
w_3	0	0	0	1	$-\frac{1}{5}$	0	$-\frac{3}{5}$	$\frac{17}{5}$

Now this is again not a valid simplex tableau; for this we must reset the framed elements in row zero:

	z	w_1	w_2	w_3	w_4	w_5	w_6	RHS
z	1	0	0	0	$-\frac{1}{5}$	0	$\frac{12}{5}$	$-\frac{88}{5}$
w_1	0	1	0	0	$\boxed{\frac{1}{5}}$	0	$-\frac{2}{5}$	$\frac{8}{5}$
w_2	0	0	1	0	$-\frac{3}{5}$	0	$\frac{1}{5}$	$\frac{16}{5}$
w_5	0	0	0	0	$-\frac{2}{5}$	1	$-\frac{1}{5}$	$\frac{49}{5}$
w_3	0	0	0	1	$-\frac{1}{5}$	0	$-\frac{3}{5}$	$\frac{17}{5}$

Second phase starts, w_4 enters the basis and w_1 leaves.

	z	w_1	w_2	w_3	w_4	w_5	w_6	RHS
z	1	1	0	0	0	0	2	-16
w_4	0	5	0	0	1	0	-2	8
w_2	0	3	1	0	0	0	-1	8
w_5	0	2	0	0	0	1	-1	13
w_3	0	1	0	1	0	0	-1	5

We get the optimal simplex tableau; end of the second phase. The optimal objective function value is -16, so the objective function value of the original (minimization) dual problem is 16. The optimal solution is $\boldsymbol{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$. The solution is unique.

e) Due to the Strong Theorem of Duality, the optimal objective function value of the primal problem (16) equals the optimal objective function value of the dual (also 16).

Using complementary slackness, further relationships can be observed: for example, it is well-known that whenever the optimal value of a dual variable is strictly positive then the corresponding constraint is tight in the primal:

$$w_2 > 0 \Rightarrow -x_1 - 2x_2 - x_3 - x_4 = -2$$

Similarly, if a constraint in the primal is not tight then the value of the corresponding dual variable is guaranteed to be zero:

$$2x_1 + x_2 + x_3 + 3x_4 < 7 \Rightarrow w_1 = 0$$

Furthermore: a positive primal variable yields that the corresponding dual constraint is tight, and contrarily, a loose (not tight) dual constraint yields that the corresponding optimal primal solution is zero:

$$x_4 > 0 \Rightarrow 3w_1 - w_2' = 3w_1 + w_2 = 8$$
$$2w_1 - w_2' = 2w_1 + w_2 > 3 \Rightarrow x_1 = 0$$

Solving Word Problems with the Simplex Algorithm

11. In a paper mill, the machines are being replaced. Two types of cardboard-cutting machines can be purchased: machine A can cut 3 boxes per one minute, one person is needed to operate it, and it costs 15,000 units of money; machine B machine can make 5 boxes per minute, but it requeres two people to supervise it, and it costs 20,000 units of money. The production plan is to produce at least 32 boxes per minute with at most 12 workers involved.

    How many A and B machines needs to be purchased to fit the production plan with minimized costs?

    a) Define the above "resource acquisition" problem as a linear program!
    b) Find an initial basis!
    c) Solve the linear program with the primal or the dual simplex algorithm!
    d) Got an integer as a result? If so, is integrality of the results guaranteed?

    Solution:

a) Mark the amount of A-type machines purchased with x_A, and mark the amount of B-type machines purchased with x_B. The workforce constraints:

$$x_A + 2x_B \leq 12 \ .$$

The plan is to produce 32 boxes, which provides the following condition:

$$3x_A + 5x_B \geq 32 \ .$$

Capital expenditures, i.e., the amount of money needed to purchase the machines:

$$15x_A + 20x_B \ .$$

The variables are non-negative. From this, the linear program:

$$\begin{array}{lrcrcr}
\min & 15x_A & + & 20x_B & & \\
\text{s.t.} & x_A & + & 2x_B & \leq & 12 \\
 & 3x_A & + & 5x_B & \geq & 32 \\
 & x_A, & & x_B & \geq & 0
\end{array}$$

b) Converting the linear program to standard form by (1) introducing slack variables to bring all conditions to a =-form, (2) inverting the second condition, and (3) rewriting the objective into maximization form:

$$\begin{array}{lrcrcrcrcr}
\max & -15x_A & - & 20x_B & & & & & & \\
\text{s.t.} & x_A & + & 2x_B & + & s_1 & & & = & 12 \\
 & -3x_A & - & 5x_B & & & + & s_2 & = & -32 \\
 & x_A, & & x_B, & & s_1, & & s_2 & \geq & 0
\end{array}$$

The slack variables form an initial unit base.

c) We will use the dual simplex algorithm. The initial simplex table:

	z	x_A	x_B	s_1	s_2	RHS
z	1	15	20	0	0	0
s_1	0	1	2	1	0	12
s_2	0	-3	$\boxed{-5}$	0	1	-32

Further iterations of the dual simplex algorithm:

	z	x_A	x_B	s_1	s_2	RHS
z	1	3	0	0	4	-128
s_1	0	$\boxed{-\frac{1}{5}}$	0	1	$\frac{2}{5}$	$-\frac{4}{5}$
x_B	0	$\frac{3}{5}$	1	0	$-\frac{1}{5}$	$\frac{32}{5}$

Finally, the optimal table:

	z	x_A	x_B	s_1	s_2	RHS
z	1	0	0	15	10	-140
x_A	0	1	0	-5	-2	4
x_B	0	0	1	3	1	4

The management of the paper mill needs to buy 4-4 machines for 140,000 units of money, and a total of $4 + 2 * 4 = 12$ workers must be employed.

d) The result an integral, but this is not guaranteed. For that we'd need to introduce an explicit integrality condition, yielding an Integer Linear Program.