Classical Applications

- Optimal Product Mix and the Resource Allocation problem
- Generalized assignment problems: continuous case, machine scheduling
- Optimal portfolio: detecting arbitrage opportunities

Optimal Product Mix

- Linear programming permeates the entire fields of economic studies, management science, operations research, and logistics
- Perhaps the most prevalent example is the Product Mix Optimization (or Production Planning) problem
- A factory has finite resources available to manufacture goods/commodities
- Products can be sold to realize immediate profit or stocked in the hope of future profit
- Goal is to determine the optimal allocation of resources in order the maximize profit
- Assumptions:
  - products and market demands are independent
  - resources are divisible
  - prices and demands can be reliably predicted

Optimal Product Mix

- Suppose that we are given
  - T: the number of business periods
  - I: the number of commodities to be produced
  - K: the number of resource types
  - a_{ik}: the amount of resource k needed to produce commodity i
  - b_{kt}: the amount of resource k available in period t
  - d_{it}: the demand for commodity i during period t
  - c_{it}: profits per unit of commodity i in period t
  - q_{it}: storage cost for commodity i in period t
- Let the variables be as follows:
  - x_{it}: the quantity of commodity i produced in period t
  - y_{it}: stock from commodity i at the end of period t

Optimal Product Mix

- The quantity of goods produced in a period plus the stock must cover the demand in each period, subject to limits on resource availability
- Goal: maximize profit while minimizing storage costs
- Let the starting stock be $y_{i0} = 0$ for each i

$$
\begin{array}{lll}
\max & \displaystyle\sum_{t=1}^{T}\sum_{i=1}^{I} c_{it}x_{it} - \sum_{t=1}^{T}\sum_{i=1}^{I} q_{it}y_{it} & \\
\text{s.t.} & y_{i,t-1} + x_{it} - y_{it} = d_{it} & i \in \{1,\ldots,I\}, t \in \{1,\ldots,T\} \\
& \displaystyle\sum_{i=1}^{I} a_{ik}x_{it} \leq b_{kt} & k \in \{1,\ldots,K\}, t \in \{1,\ldots,T\} \\
& x_{it}, y_{it} \geq 0 & i \in \{1,\ldots,I\}, t \in \{1,\ldots,T\}
\end{array}
$$

Optimal Product Mix

- There are endless variants of the product mix problem
  - demands may be delayed to a later period
  - resources may be substitutable for one another
  - or may be stored in the inventory across periods
- Simplest case: plan for a single period only
- No inventory needs to be planned in this case
- Suppose that market demands are infinite
- Then, the amount of goods produced is limited only by resource availability and profitability

Optimal Product Mix

$$
\begin{aligned}
\max \sum_{i=1}^{I} c_i x_i & \\
\text{s.t.} \sum_{i=1}^{I} a_{ik} x_i \le b_k & \qquad i \in \{1, \ldots, I\} \\
x_i \ge 0 & \qquad i \in \{1, \ldots, I\}
\end{aligned}
$$

- This is a “classical” Resource Allocation problem
- The simplex method was invented for these types of problems
- Parameters c_i, b_k, and a_{ik} are nonnegative
- After bringing to standard form by introducing slack variables: trivial primal feasible initial basis on the slacks

Optimal Product Mix: Example

- The ACME Winery produces three types of wine: white, red and cuvée, from three types of grape (G1, G2, and G3)
  - 2 tonnes of G1 grape and 1 ton of G2 grape is needed to produce one barrel of white wine
  - 2 tonnes of G3 grape is needed for one barrel of red wine
  - and finally for a barrel of cuvée wine one tonne from each type of grape is used
- Availability of grape: 8 tonnes from G1, 4 tonnes from G2, and 6 tonnes of G3 grape
- The profit is 31 thousand USD per barrel of white wine, 22 thousand USD on red wine, and 35 thousand USD on cuvée
- **Question:** How many barrels to produce from each type of wine in order to maximize profits?

Optimal Product Mix: Example

- Collect available data into a table as follows:

	Demand [t/b]			Profit [th USD/b]
	G1	G2	G3	
White wine	2	1		31
Red wine			2	22
Cuvée wine	1	1	1	35
Capacity [t]	8	4	6	

- Let x_1, x_2, and x_3 denote the amount of wine produced from white, red, and cuvée wines ([barrels, b])

$$
\begin{array}{lrcrcrcl}
\max & 31x_1 & + & 22x_2 & + & 35x_3 & & \\
\text{s.t.} & 2x_1 & & & + & x_3 & \leq & 8 \\
 & x_1 & & & + & x_3 & \leq & 4 \\
 & & & 2x_2 & + & x_3 & \leq & 6 \\
 & x_1, & & x_2, & & x_3 & \geq & 0
\end{array}
$$

Optimal Product Mix: Example

- Trivial primal feasible initial basis on the columns of the slack variables: primal simplex

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	-31	-22	-35	0	0	0	0
x_4	0	2	0	1	1	0	0	8
x_5	0	1	0	$\boxed{1}$	0	1	0	4
x_6	0	0	2	1	0	0	1	6

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	4	-22	0	0	35	0	140
x_4	0	1	0	0	1	-1	0	4
x_3	0	1	0	1	0	1	0	4
x_6	0	-1	$\boxed{2}$	0	0	-1	1	2

Optimal Product Mix: Example

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	-7	0	0	0	24	11	162
x_4	0	$\boxed{1}$	0	0	1	-1	0	4
x_3	0	1	0	1	0	1	0	4
x_2	0	$-\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	1

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	0	0	7	17	11	190
x_1	0	1	0	0	1	-1	0	4
x_3	0	0	0	1	-1	2	0	0
x_2	0	0	1	0	$\frac{1}{2}$	-1	$\frac{1}{2}$	3

- The optimal mix is 4 barrels of white wine, 3 barrels of red wine, and no cuvée, the profit is 190 thousand USD

Optimal Product Mix: Example

- Based on market analysis, the demand for cuvée will increase in the following year and so the price per barrel for cuvée wine is expected to grow to 44 thousand USD while the price for the rest of the wines remains fixed
- **Question:** How to change the product mix to maximize profits, given the anticipated price changes?
- Sensitivity analysis: the objective function changes
- In particular, the objective coefficient for a basic variable x_3 changes in the optimal tableau: $c_3 = 35 \to c_3' = 44$
- Add the row of x_3 (the second row!) to row zero exactly $c_3' - c_3 = 9$ times
- Take note to fix the reduced cost for x_3 in the objective row at zero
- The resultant tableau is not primal optimal: primal simplex

Optimal Product Mix: Example

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	0	0	-2	35	11	190
x_1	0	1	0	0	$\boxed{1}$	-1	0	4
x_3	0	0	0	1	-1	2	0	0
x_2	0	0	1	0	$\frac{1}{2}$	-1	$\frac{1}{2}$	3

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	2	0	0	0	33	11	198
x_4	0	1	0	0	1	-1	0	4
x_3	0	1	0	1	0	1	0	4
x_2	0	$-\frac{1}{2}$	1	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	1

- The product mix changes to 1 barrel of red wine and 4 barrels of cuvée
- 4 tonnes of G1 grape is surplus (since $x_4 = 4$)

Optimal Product Mix: Example

- **Question:** How should the price of white wine change in order for it to become profitable to produce? What is the optimal product mix if the ACME Winery produces 2 barrels of white wine at this price tag?
- The linear program in the space of the nonbasic variable x_1

$$
\begin{aligned}
\max \quad & 198 - 2x_1 \\
\text{s.t.} \quad & \begin{bmatrix} x_4 \\ x_3 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -\frac{1}{2} \end{bmatrix} x_1 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

- If the price increased by 2 thousand USD the objective (the profit) would no longer decrease with increasing x_1
- If 2 barrels of white wine was produced at this price, then cuvée output would drop to two tonnes and the remaining G3 grape would be enough to one barrel of red wine

Generalized Assignment Problem

- Another crucial application of linear programming in operations research
- Given m agents and n jobs that must be assigned to agents
- Each task can be performed by each agent, but with different effectiveness
  - agent i can do job j with w_{ij} units of effort, meanwhile we realize p_{ij} profits
  - agent i has w_i units of working capacity
  - b_j units of job j must be done
- Goal is to maximize profits
- Suppose that jobs are arbitrarily divisible
- Otherwise we get an integer linear program: NP-hard

Generalized Assignment Problem

- Let x_{ij} denote the quantity of job j performed by agent i [units]

$$
\begin{aligned}
\max \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} x_{ij} & \\
\sum_{j=1}^{n} w_{ij} x_{ij} \leq w_i & \qquad i \in \{1, \ldots, m\} \\
\sum_{i=1}^{m} x_{ij} = b_j & \qquad j \in \{1, \ldots, n\} \\
x_{ij} \geq 0 & \qquad i \in \{1, \ldots, m\}, \\
& \qquad j \in \{1, \ldots, n\}
\end{aligned}
$$

Machine Scheduling

- The ACME Steel Factory manufactures small, medium and large steel beams. The factory uses two types of machines: A, and B, with capacity to produce the below quantities of different types of steel beams per hour:

Beam	Machine [unit/h]		Demand [unit/week]
	A	*B*	
Small	3	6	96
Medium size	2	4	96
Large	2	3	72

- The table specifies the weekly demand for each type of beam as well, machine time available is 40 hours per week per machine and the machines cost at 8 (A) and 4 (B) thousand USD per hour be operated
- **Task:** Design an optimal schedule that minimizes operations costs while covering the demands

Machine Scheduling

- Denote by x_{1A} the quantity of small, by x_{2A} the quantity of medium, and by x_{3A} the quantity of large beams produced by machine A [units]; similar to machine B
- Machine A spends $\frac{x_{1A}}{3}$ hours to produce small beams
- This is a special type of continuous assignment problem

$$
\begin{aligned}
\min \quad & 8\left(\tfrac{x_{1A}}{3} + \tfrac{x_{2A}}{2} + \tfrac{x_{3A}}{2}\right) + 4\left(\tfrac{x_{1B}}{6} + \tfrac{x_{2B}}{4} + \tfrac{x_{3B}}{3}\right) \\
& \tfrac{x_{1A}}{3} + \tfrac{x_{2A}}{2} + \tfrac{x_{3A}}{2} \leq 40 \\
& \tfrac{x_{1B}}{6} + \tfrac{x_{2B}}{4} + \tfrac{x_{3B}}{3} \leq 40 \\
& x_{1A} + x_{1B} = 96 \\
& x_{2A} + x_{2B} = 96 \\
& x_{3A} + x_{3B} = 72 \\
& x_{1A}, x_{2A}, x_{3A} \geq 0 \\
& x_{1B}, x_{2B}, x_{3B} \geq 0
\end{aligned}
$$

Machine Scheduling

- A simpler linear program can be obtained by letting x_{1A} to rather denote the amount of machine hours spent by machine A on small beams, and similar for the rest of beam types/machines

$$
\begin{aligned}
\min \quad 8(x_{1A} + x_{2A} + x_{3A}) + 4(x_{1B} + x_{2B} + x_{3B}) \\
x_{1A} + x_{2A} + x_{3A} \leq 40 \\
x_{1B} + x_{2B} + x_{3B} \leq 40 \\
3x_{1A} + 6x_{1B} = 96 \\
2x_{2A} + 4x_{2B} = 96 \\
2x_{3A} + 3x_{3B} = 72 \\
x_{1A}, x_{2A}, x_{3A} \geq 0 \\
x_{1B}, x_{2B}, x_{3B} \geq 0
\end{aligned}
$$

- May still have fractional units of beams in the schedule: price paid for the simplicity of the continuous model

Machine Scheduling

- Introduce slacks x_7 and x_8 to bring to standard form
- Introduce artificial variables x_9, x_{10}, and x_{11} to find the starting basis
- Phase One: still invalid tableau (again omit the column of z)
- Zero out the objective row elements marked in red (nonzero reduced costs in basic columns) to get a valid tableau

	x_{1A}	x_{1B}	x_{2A}	x_{2B}	x_{3A}	x_{3B}	x_7	x_8	x_9	x_{10}	x_{11}	RHS
z	0	0	0	0	0	0	0	0	$\boxed{1}$	$\boxed{1}$	$\boxed{1}$	0
x_7	1	0	1	0	1	0	1	0	0	0	0	40
x_8	0	1	0	1	0	1	0	1	0	0	0	40
x_9	3	6	0	0	0	0	0	0	$\boxed{1}$	0	0	96
x_{10}	0	0	2	4	0	0	0	0	0	$\boxed{1}$	0	96
x_{11}	0	0	0	0	2	3	0	0	0	0	$\boxed{1}$	72

Machine Scheduling

- The optimal simplex tableau for Phase One

	x_{1A}	x_{1B}	x_{2A}	x_{2B}	x_{3A}	x_{3B}	x_7	x_8	x_9	x_{10}	x_{11}	RHS
z	0	0	0	0	0	0	0	0	1	1	1	0
x_{3A}	0	0	0	0	1	0	-3	-6	1	$\frac{3}{2}$	2	24
x_{2B}	0	0	$\frac{1}{2}$	1	0	0	0	0	0	$\frac{1}{4}$	0	24
x_{3B}	0	0	0	0	0	1	2	4	$-\frac{2}{3}$	-1	-1	8
x_{1A}	1	0	1	0	0	0	4	6	-1	$-\frac{3}{2}$	-2	16
x_{1B}	0	1	$-\frac{1}{2}$	0	0	0	-2	-3	$\frac{2}{3}$	$\frac{3}{4}$	1	8

- Objective value is 0: the basis obtained is feasible for the original problem
- Artificial variables have left the basis: remove the corresponding columns

Machine Scheduling

- The original problem was minimization
  - invert row zero twice (once for the $\min \to \max$ conversion and once when writing into the tableau)
  - take note that the resultant objective value will need to be inverted at the end
- Phase Two: still not a valid tableau, zero out the reduced costs marked in red

	x_{1A}	x_{1B}	x_{2A}	x_{2B}	x_{3A}	x_{3B}	x_7	x_8	RHS
z	$\boxed{8}$	$\boxed{4}$	8	$\boxed{4}$	$\boxed{8}$	$\boxed{4}$	0	0	0
x_{3A}	0	0	0	0	$\boxed{1}$	0	-3	-6	24
x_{2B}	0	0	$\frac{1}{2}$	$\boxed{1}$	0	0	0	0	24
x_{3B}	0	0	0	0	0	$\boxed{1}$	2	4	8
x_{1A}	$\boxed{1}$	0	1	0	0	0	4	6	16
x_{1B}	0	$\boxed{1}$	$-\frac{1}{2}$	0	0	0	-2	-3	8

Machine Scheduling

- Phase Two: optimal simplex tableau

	x_{1A}	x_{1B}	x_{2A}	x_{2B}	x_{3A}	x_{3B}	x_7	x_8	RHS
z	2	0	2	0	0	0	0	8	-448
x_{3A}	$\frac{3}{4}$	0	$\frac{3}{4}$	0	1	0	0	$-\frac{3}{2}$	36
x_{2B}	0	0	$\frac{1}{2}$	1	0	0	0	0	24
x_7	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	0	1	$\frac{3}{2}$	4
x_{3B}	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	1	0	1	0
x_{1B}	$\frac{1}{2}$	1	0	0	0	0	0	0	16

- Schedule 0 and 16 hours for small, 0 and 24 hours for medium, and 36 and 0 hours for producing large beams on machine A and B, respectively
- Cost is 448 thousand USD, the schedule for machine A contains 4 hours of dead time, machine B operates full time

Machine Scheduling

- The management of the ACME Steel Factory decides to rationalize the work schedule
- Since machine B operates full time, the management decides to introduce a new work shift for machine B
- **Task:** Compute the optimal schedule and cost as the function of extra hours of work of machine B
- Parametric analysis when perturbing the RHS: denote the available work span for machine B be $b_B = 40 + \lambda$
- Determine the cost and the optimal schedule in terms of λ

$$\bar{\boldsymbol{b}}' = \boldsymbol{B}^{-1}\boldsymbol{b}' = \boldsymbol{B}^{-1}\boldsymbol{e}_2 = (\boldsymbol{B}^{-1})_2$$

where $(\boldsymbol{B}^{-1})_2$ denotes the second column of $\boldsymbol{B}^{-1}$

Machine Scheduling

- We would need to compute $\boldsymbol{B}^{-1}$, or at least the second column
- Observe that in the original problem the column that belong to the slack variable x_8 is exactly $\boldsymbol{e}_2$
- Thus we can always read the second row of the current basis from the column of x_8 in the simplex tableau

$$\bar{\boldsymbol{b}}' = (\boldsymbol{B}^{-1})_2 = \boldsymbol{y}_2 = \begin{bmatrix} -\frac{3}{2} \\ 0 \\ \frac{3}{2} \\ 1 \\ 0 \end{bmatrix}$$

- Hence: $S = \{1\}$, $r = 1$, and $\bar{\lambda} = -\frac{\bar{b}_1}{\bar{b}'_1} = -\frac{36}{-\frac{3}{2}} = 24 = \lambda_1$

Machine Scheduling

- The current basis is optimal for $\lambda \in [0, 24]$

$$z(\lambda) = \boldsymbol{c_B}^T(\bar{\boldsymbol{b}} + \lambda \bar{\boldsymbol{b}}') = -448 + 8\lambda$$

$$\boldsymbol{x}(\lambda) = \begin{bmatrix} x_{3A} \\ x_{2B} \\ x_7 \\ x_{3B} \\ x_{1B} \end{bmatrix} = \begin{bmatrix} 36 \\ 24 \\ 4 \\ 0 \\ 16 \end{bmatrix} + \begin{bmatrix} -\frac{3}{2} \\ 0 \\ \frac{3}{2} \\ 1 \\ 0 \end{bmatrix} \lambda$$

- If the work span of machine B was increased to 64 hours than all production would move to this machine
- Meanwhile, the cost decreases as $448 - 8\lambda = 256$ [th USD]
- End of parametric analysis: after the dual simplex pivot the resultant basis is optimal for any $\lambda \geq 24$

Arbitrage Pricing

- Given n instruments/securities/investment options, traded according to a **discrete market model**
- The market can be in one of m possible eventual states at the end of the trading period: the profit on instrument j in the eventual state i is r_{ij} (can be negative)
- There are two possible investment strategies:
  - long position: we buy instrument j at the beginning of the investment period which we sell at the end
  - "short" position: we sell instrument j at the beginning of the period that we are confined to buy back at the end
- **Arbitrage opportunity:** a portfolio of long and short positions that produces positive expected income
- Such portfolio will bring net profit at no risk of loss
- **Question:** is there an arbitrage opportunity at the market?

Arbitrage Pricing

- Let x_j denote the weight of instrument j in the portfoilo
- Long position ($x_j > 0$): we receive $(1 + r_{ij})x_j$ amount at the end, provided the market outcome is i
- Net profit is $r_{ij}x_j$, we bet for price increase
- Short position ($x_j < 0$): first we get $|x_j|$ amount and then we buy at $(1 + r_{ij})|x_j|$ amount at the end, net profit: $r_{ij}x_j$
- Positive income if $r_{ij} < 0$, we bet for price reduction
- Let $\boldsymbol{R} = [r_{ij}]$ be the $m \times n$ **payoff matrix**
- The net income for market outcome i: $\sum_j r_{ij}x_j$
- Arbitrage if the net profit for a portfolio is positive for every possible market outcome

$$\exists \boldsymbol{x} : \boldsymbol{Rx} > \mathbf{0}$$

Arbitrage Pricing: Example

- Suppose that there are $n = 2$ instruments traded on the market, which can be of $m = 3$ possible states at the end of the investment period
- The payoff for each instrument/outcome combination is as follows:

$$\boldsymbol{R} = \begin{bmatrix} -\frac{1}{4} & \frac{1}{8} \\ -\frac{1}{4} & 0 \\ 0 & \frac{1}{5} \end{bmatrix}$$

- For instance, the first instrument will lose 25% in the first two outcomes and produces zero profit otherwise
- Let the portfolio be $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, where x_1 is the position on the first and x_2 on the second instrument
- Long position: $x_i > 0$, short position: $x_i < 0$

Arbitrage Pricing: Example

- For the first market outcome, the net profit is $-\frac{1}{4}x_1 + \frac{1}{8}x_2$, positive income if this is positive
- For instance, if we maintain 1 unit of short position on the first instrument ($x_1 = -1$) and one unit if long position on the second ($x_2 = 1$), then the profit is $\frac{3}{8}$
- Same portfolio is profitable for the second outcome as well (net profit: $\frac{1}{4}$)
- What is more, it is profitable for the third market outcome too (net profit: $\frac{1}{5}$)
- We have found an arbitrage opportunity: there is a portfolio that brings positive income for every possible market outcome (i.e., at zero risk)
- Undesirable on “normal” markets

Arbitrage Pricing

- Arbitrage opportunity if $\exists \boldsymbol{x} : \boldsymbol{R}\boldsymbol{x} > \mathbf{0}$
- We use duality theory to characterize such opportunities
- "$>$" type of constraints are hard to handle in linear programs
- Consider the below linear program instead:

$$\begin{aligned} \min \quad & \mathbf{0}\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{R}\boldsymbol{x} \geq \mathbf{0} \end{aligned}$$

- Let $\boldsymbol{p}^T$ be the vector of dual variables for the m primal constraints
- The dual linear program:

$$\begin{aligned} \max \quad & \boldsymbol{p}^T\mathbf{0} \\ \text{s.t.} \quad & \boldsymbol{p}^T\boldsymbol{R} = \mathbf{0} \\ & \boldsymbol{p}^T \geq \mathbf{0} \end{aligned}$$

Arbitrage Pricing

- For both linear program it holds that any feasible solution is immediately optimal as well
- Trivial solution: $\boldsymbol{x} = \mathbf{0}$ and $\boldsymbol{p}^T = \mathbf{0}$
- We seek for nontrivial primal solutions $\boldsymbol{x}$ for which $\boldsymbol{R}\boldsymbol{x} > \mathbf{0}$
- We rather study nontrivial solutions of the dual
- **Fundamental Theorem of Asset Pricing:** there exists $\boldsymbol{x}$: $\boldsymbol{R}\boldsymbol{x} > \mathbf{0}$, if and only if there is no $\boldsymbol{p}^T$ that satisfies the below:

$$\boldsymbol{p}^T\boldsymbol{R} = \mathbf{0}, \quad \boldsymbol{p}^T \geq \mathbf{0}, \quad \boldsymbol{p}^T \neq \mathbf{0}$$

- **Proof:** Farkas lemma $\square$
- The reverse is also true: if there is $\boldsymbol{p}^T$: $\boldsymbol{p}^T\boldsymbol{R} = \mathbf{0}$, $\boldsymbol{p}^T \geq \mathbf{0}$, $\boldsymbol{p}^T \neq \mathbf{0}$ then there is no arbitrage on the market

Arbitrage Pricing

- This result could use some explanation
- Suppose there is $\boldsymbol{p}^T \neq \mathbf{0}$, $\boldsymbol{p}^T \geq \mathbf{0}$ according to the conditions of the theorem
- Normalize so that $\boldsymbol{p}^T\mathbf{1} = 1$ (can always be done)
- We may interpret components of $\boldsymbol{p}^T$ as a probability on the outcomes
- Then, by $\boldsymbol{p}^T\boldsymbol{R} = \mathbf{0}$ for any portfolio $\boldsymbol{x}$ the expected net profit is $\mathbb{E}(\text{profit}) = \boldsymbol{p}^T\boldsymbol{R}\boldsymbol{x} = 0$
- The Fundamental Theorem states that there is *no* arbitrage in the market if there is a probability distribution on market outcomes with zero expected net profit for *any* portfolio
- Such $\boldsymbol{p}^T$ probabilities are called the **risk-neutral probabilities** (completely independent from the physical probabilities of the states)

Arbitrage Pricing: Example

- There are two instruments on a market, which belong to competitive enterprises: if the price of one of the instruments increases then the other decreases, and vice versa
- It is also possible that the price of the first instrument remains the same and the second brings some minimal profit

$$\boldsymbol{R} = \begin{bmatrix} -\frac{1}{5} & \frac{3}{10} \\ 0 & \frac{1}{10} \\ \frac{1}{5} & -\frac{1}{5} \end{bmatrix}$$

- **Question:** is there an arbitrage opportunity in the market? If yes, which one is the optimal portfolio?

Arbitrage Pricing: Example

- Find portfolio $\boldsymbol{x}$ so that $\boldsymbol{Rx} > \mathbf{0}$
- If there is such $\boldsymbol{x}$ then every other $\lambda \boldsymbol{x}$ is a solution for $\lambda > 0$ and hence $\boldsymbol{Rx}$ can be made arbitrarily large
- It is enough the solve the below linear program:

$$\begin{array}{ll} \max & \mathbf{0}\boldsymbol{x} \\ & \boldsymbol{Rx} \geq \mathbf{1} \end{array}$$

- Let $\boldsymbol{x} = \boldsymbol{x}^+ - \boldsymbol{x}^-$ and let $\boldsymbol{x}_s$ be slack variables

$$\begin{array}{lrcrcrcr} \max & \mathbf{0}\boldsymbol{x}^+ & + & \mathbf{0}\boldsymbol{x}^- & + & \mathbf{0}\boldsymbol{x}_s & & \\ \text{s.t.} & -\boldsymbol{R}\boldsymbol{x}^+ & + & \boldsymbol{R}\boldsymbol{x}^- & + & \boldsymbol{x}_s & = & -\mathbf{1} \\ & \boldsymbol{x}^+, & & \boldsymbol{x}^-, & & \boldsymbol{x}_s, & \geq & \mathbf{0} \end{array}$$

- The columns for $\boldsymbol{x}_s$ form a dual feasible initial basis: dual simplex

Arbitrage Pricing: Example

- The initial simplex tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	0	0	0	0	0	0	0	0
x_5	$\frac{1}{5}$	$-\frac{3}{10}$	$-\frac{1}{5}$	$\frac{3}{10}$	1	0	0	-1
x_6	0	$-\frac{1}{10}$	0	$\frac{1}{10}$	0	1	0	-1
x_7	$-\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$-\frac{1}{5}$	0	0	1	-1

- The optimal tableau:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	0	0	0	0	0	0	0	0
x_2	0	1	0	-1	-10	0	-10	20
x_6	0	0	0	0	-1	1	-1	1
x_1	1	0	-1	0	-10	0	-15	25

Arbitrage Pricing: Example

- We have found an arbitrage opportunity: maintain a long position of 25 units from the first and a long position of 20 units on the second instrument
- At the end of the period the expected net profit: $\boldsymbol{Rx} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$
- If one component of $\boldsymbol{R}$ changes: $\boldsymbol{R} = \begin{bmatrix} -\frac{1}{5} & \frac{3}{10} \\ 0 & \frac{1}{10} \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix}$
- No arbitrage opportunity
- Dual argumentation: if in this case the individual market outcomes occur with probability $\boldsymbol{p}^T = [\frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3}]$, then every portfolio is “fair” (produces zero profit/loss)