Starting Solution and Analysis: A Summary

*WARNING: this is just a summary of the material covered in the full slide-deck **Starting Solution and Analysis** that will orient you as per the topics covered there; you are required to learn the full version, not just this summary!*

- Finding an initial basic feasible solution: the Artificial Variable technique
- Sensitivity analysis: the effect of changing the objective function
- Parametric analysis: perturbation of the Right-Hand-Side (optional)

Recall: The Simplex Algorithm

- We need an initial basic feasible solution to start the simplex
- In canonical form an initial basis is easy to find
- **Maximization problem:** $\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$
- Into standard form: $\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} + \boldsymbol{I}\boldsymbol{x}_s = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$
- If $\boldsymbol{b} \geq \boldsymbol{0}$ then the slack variables constitute a primal feasible initial basis: **primal simplex**
- For a **minimization problem** in canonical form:

$$\min\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$$

- Dual feasible initial basis on the columns of the slacks if $\boldsymbol{c}^T \geq \boldsymbol{0}^T$: **dual simplex**
- If neither case occurs then the simplex cannot be started: need a generic way for finding initial basic feasible solutions

Starting the Simplex Method

- Find an initial basic feasible solution for the linear program given in standard form:

$$
\begin{aligned}
z = \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\
& \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned}
$$

  where $\boldsymbol{A}$ is $m \times n$ with $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}, \boldsymbol{b}) = m$, $\boldsymbol{b}$ is a column m, $\boldsymbol{x}$ a column n, and $\boldsymbol{c}^T$ is a row n-vector
- Suppose furthermore that $\boldsymbol{b} \geq \boldsymbol{0}$ (if there is row i with $b_i < 0$, then invert the row to get $-b_i > 0$)

The Artificial Variable Technique

- Introduce $\boldsymbol{x_a}$ artificial variables and consider the modified linear program:

$$\begin{aligned} z = \min \quad & \mathbf{1}^T \boldsymbol{x_a} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{x_a} = \boldsymbol{b} \\ & \boldsymbol{x}, \boldsymbol{x_a} \geq \mathbf{0} \end{aligned}$$

  where $\mathbf{1}^T$ is a row vector (of proper size) with all components set to 1

- There is a trivial initial basis for the modified problem
- Since the columns of $\boldsymbol{x_a}$ form an identity matrix, we have a feasible initial basis on $\boldsymbol{x}_a$: $\boldsymbol{B} = \boldsymbol{I}$ and $\boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{b} \geq \mathbf{0}$ by assumption

The Artificial Variable Technique

- Solve the modified problem from the initial basis defined by the artificial variables
- The optimum is $z_0 = \mathbf{1}^T \boldsymbol{x_a}$ (the sum of the artificial variables in the solution)
- **Thoerem:** if $z_0 > 0$ then the original linear program is infeasible
- If, on the other hand, $z_0 = 0$, then $\boldsymbol{x_a} = \mathbf{0}$
- In this case the original linear program is feasible
- Solve it from the resultant basis

The Two–Phase Simplex Method

- **Phase One:** find an initial basis
- Solve the modified linear program augmented with the artificial variables $\boldsymbol{x_a}$

$$\begin{aligned} z = \max \quad & -\mathbf{1}^T \boldsymbol{x_a} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{x_a} = \boldsymbol{b} \\ & \boldsymbol{x}, \boldsymbol{x_a} \geq \mathbf{0} \end{aligned}$$

- If $\boldsymbol{x_a} \neq \mathbf{0}$ then the linear program is infeasible
- Otherwise, $\boldsymbol{x_a} = \mathbf{0}$ and suppose that all artificial variables have left the basis
- If not, the remaining artificial variables must be “pivoted” out from the basis manually, we do not discuss this here
- **Phase Two:** remove the artificial variables, restore the original objective function and run the simplex from the current basis

The Two–Phase Simplex: Example

- Solve the linear program using the Two–Phase Simplex

$$\begin{array}{lrcrcr} \min & -3x_1 & + & 4x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \leq & 4 \\ & 2x_1 & + & 3x_2 & \geq & 18 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

- Convert to maximization and bring to standard form by introducing slack variables (take note of the "$\geq$" type of constraints and that eventually we need $\boldsymbol{b} \geq \mathbf{0}$!)

$$\begin{array}{lrcrcrcrcr} \max & 3x_1 & - & 4x_2 & & & & & & \\ \text{s.t.} & x_1 & + & x_2 & + & x_3 & & & = & 4 \\ & 2x_1 & + & 3x_2 & & & - & x_4 & = & 18 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

The Two–Phase Simplex: Example

- No trivial primal or dual feasible basis
- Introduce artificial variables: it is enough add an artificial variable x_5 to the second row
- This, together with the slack variable x_3, will provide a proper initial (identity) basis
- Solve the below linear program as the first phase:

$$
\begin{array}{lrcrcrcrcrcr}
\max & & & & & & & & & -x_5 & & \\
\text{s.t.} & x_1 & + & x_2 & + & x_3 & & & & & = & 4 \\
& 2x_1 & + & 3x_2 & & & - & x_4 & + & x_5 & = & 18 \\
& x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
\end{array}
$$

- Note that we have converted the objective to maximization: will need to invert the resultant objective value!

The Two–Phase Simplex: Example

- Initial basis: $\boldsymbol{B} = [\boldsymbol{a_3} \quad \boldsymbol{a_5}]$, $\boldsymbol{c_B}^T = [0 \quad -1]$, $\boldsymbol{c_N}^T = \boldsymbol{0}$
- Not a valid simplex tableau yet: there is a nonzero element in the objective row for the basic variable x_5
  - "pivot": subtract the row of x_5 from row 0

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	0	0	0	1	0
x_3	0	1	1	1	0	0	4
x_5	0	2	3	0	–1	1	18

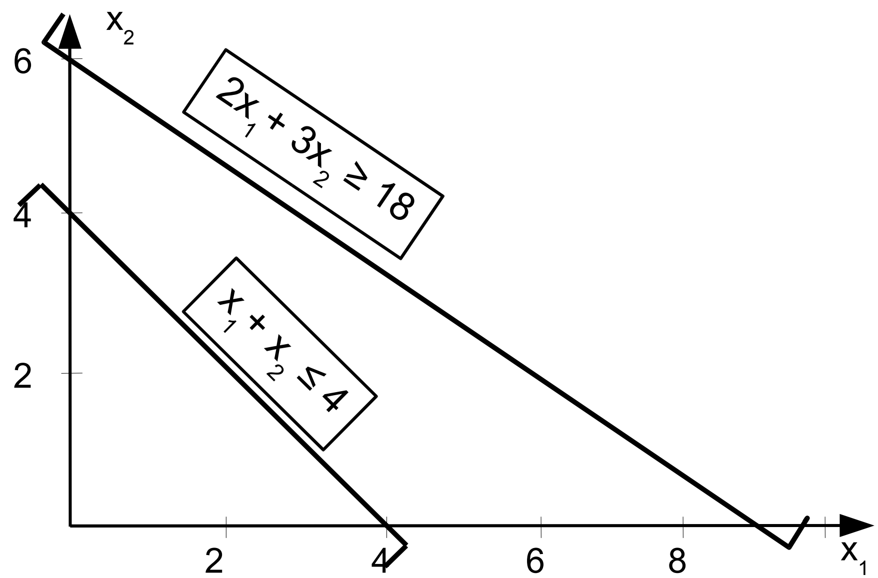
	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	–2	–3	0	1	0	–18
x_3	0	1	1	1	0	0	4
x_5	0	2	3	0	–1	1	18

The Two–Phase Simplex: Example

- We get an optimal tableau after the pivot, with optimal objective function value -6 (do not forget to invert this!)

	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	3	1	0	-6
x_2	1	1	1	0	0	4
x_5	-1	0	-3	-1	1	6

- Since $\min x_5 = 6$, the artificial variable could not be eliminated
- The original linear program is infeasible



Sensitivity Analysis

- Linear programs are often used to model real problems whose parameters are uncertain or subject to measurement errors or noise
- In a Resource Allocation problem, for instance, the estimated prices might be uncertain, capacities might be expanded by investing into new equipment, etc.
- **Question:** how does the optimal solution of a linear program $\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ depend on the perturbation of the input parameters?
    - here we discuss only the case when the objective function coefficients $\boldsymbol{c}^T$ change
    - sensitivity analysis goes similarly for the cases when the RHS vector $\boldsymbol{b}$ or the constraint matrix $\boldsymbol{A}$ change
- The idea is that we do not want to re-optimize the changed linear program from scratch

Changing the Objective Function

- Let $\boldsymbol{B}$ be an optimal basis for the linear program $\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$
- We characterize the change in the optimal solution $\boldsymbol{x}$ and the optimal objective function value when the k-th objective function coefficient c_k is changed to c'_k
- The simplex tableau of the original linear program in the basis $\boldsymbol{B}$

	z	$\boldsymbol{x}_{\boldsymbol{B}}$	$\boldsymbol{x}_{\boldsymbol{N}}$	RHS	
z	1	$\boldsymbol{0}$	$\boldsymbol{c}_{\boldsymbol{B}}{}^T\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c}_{\boldsymbol{N}}{}^T$	$\boldsymbol{c}_{\boldsymbol{B}}{}^T\boldsymbol{B}^{-1}\boldsymbol{b}$	row 0
$\boldsymbol{x}_{\boldsymbol{B}}$	$\boldsymbol{0}$	$\boldsymbol{I}_m$	$\boldsymbol{B}^{-1}\boldsymbol{N}$	$\boldsymbol{B}^{-1}\boldsymbol{b}$	row 1...m

- $\boldsymbol{B}$ is (primal) feasible if $\boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$
- $\boldsymbol{B}$ is (primal) optimal if $\boldsymbol{c}_{\boldsymbol{B}}{}^T\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c}_{\boldsymbol{N}}{}^T \geq \boldsymbol{0}$

Changing the Objective Function

1.) The changed objective coefficient c_k belongs to a nonbasic variable $x_k : k \in N$

$$\boldsymbol{c_N}^T \to (\boldsymbol{c'_N})^T = \boldsymbol{c_N} + (c'_k - c_k)\boldsymbol{e_k}^T$$

- In this case no change occurs in rows $1, \ldots, m$ of the simplex tableau, only the objective row (row 0) changes

$$\boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c_N}^T \to \boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} - (\boldsymbol{c_N}')^T = \boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c_N}^T - (c'_k - c_k)\boldsymbol{e_k}^T$$

- In fact, only the reduced cost z_k for the nonbasic variable x_k changes:

$$z_k \to z'_k = z_k - (c'_k - c_k)$$

Changing the Objective Function

- If $z_k - (c'_k - c_k) \geq 0$ then basis $\boldsymbol{B}$ remains optimal
- For instance, if we **reduce** the cost of a nonbasic variable the current basis is guaranteed to remain optimal
- The objective function value does not change (x_k remains at 0)
- If, on the other hand, $z_k - (c'_k - c_k) < 0$, then basis $\boldsymbol{B}$ is no longer optimal according to the changed objective
- Run the primal simplex from basis $\boldsymbol{B}$ to obtain the new optimum
- Using this method we do not need to re-run the Two–Phase simplex from scratch, rather the simplex method continues from the optimal basis of the original problem
- This is the idea in sensitivity analysis

Changing the Objective Function

2.) The changed objective coefficient c_k belongs to a basic variable $x_k : k \in B$

- Let x_k be the t-th basic variable: $x_k \equiv x_{B_t}$

$$\boldsymbol{c_B}^T \to (\boldsymbol{c'_B})^T = \boldsymbol{c_B} + (c'_{B_t} - c_{B_t})\boldsymbol{e_t}^T$$

- Again, only the objective row changes in the tableau
- Basic variables (including x_{B_t}) still have zero reduced cost
- The reduced costs for nonbasic variables change, the j-th:

$$z'_j = (\boldsymbol{c'_B})^T \boldsymbol{B}^{-1}\boldsymbol{a}_j - c_j = \boldsymbol{c_B}^T \boldsymbol{B}^{-1}\boldsymbol{a}_j - c_j + \\ [0 \quad 0 \; \ldots \; c'_{B_t} - c_{B_t} \; \ldots \quad 0]\boldsymbol{y_j} = z_j + (c'_{B_t} - c_{B_t})y_{tj}$$

- Add $c'_{B_t} - c_{B_t}$ times the row of x_{B_t} to row 0
- Then zero out the reduced cost for x_{B_t}

Changing the Objective: Example

- Solve the below linear program:

$$\begin{array}{lccccccc} \max & 2x_1 & - & x_2 & + & x_3 & & \\ \text{s.t.} & x_1 & + & x_2 & + & x_3 & \leq & 6 \\ & -x_1 & + & 2x_2 & & & \leq & 4 \\ & x_1, & & x_2, & & x_3 & \geq & 0 \end{array}$$

- The slack variables form a feasible initial basis

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	−2	1	−1	0	0	0
x_4	0	$\boxed{1}$	1	1	1	0	6
x_5	0	−1	2	0	0	1	4

Changing the Objective: Example

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	3	1	2	0	12
x_1	0	1	1	1	1	0	6
x_5	0	0	3	1	1	1	10

- Optimal tableau, with basic variables $B = \{1, 5\}$
- Reduce $c_2 = -1$ to $c'_2 = -3$: since x_2 is not basic only the reduced cost z_2 changes in row 0:
  $z'_2 = z_2 - (c'_2 - c_2) = 3 - (-3 - (-1)) = 5$

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	5	1	2	0	12
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

- The tableau remains optimal, the objective function value does not change

Changing the Objective: Example

- If now the objective coefficient for x_2 is changed to $c_2' = 3$, then $z_2' = -1$
- The resultant tableau is no longer optimal: primal simplex

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	-1	1	2	0	12
x_1	0	1	1	1	1	0	6
x_5	0	0	$\boxed{3}$	1	1	1	10

- The optimal tableau

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	0	$\frac{4}{3}$	$\frac{7}{3}$	$\frac{1}{3}$	$\frac{46}{3}$
x_1	0	1	0	$\frac{2}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{8}{3}$
x_2	0	0	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{10}{3}$

Changing the Objective: Example

- Now change the cost for a basic variable, say, x_1, from $c_1 = 2$ to zero
- The optimal tableau of the original problem:

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	3	1	2	0	12
x_1	0	1	1	1	1	0	6
x_5	0	0	3	1	1	1	10

- Add the first row to row 0 exactly $c'_1 - c_1 = -2$ times (that is, subtract the double)

Changing the Objective: Example

- Performing the row operation, the objective value changes:

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	-2	1	-1	0	0	0
x_1	0	1	1	1	1	0	6
x_5	0	0	3	1	1	1	10

- Since only the elements that belong to nonbasic variables need to be altered in the objective row, we simply set the reduced cost for x_1 to zero

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	1	-1	0	0	0
x_1	0	1	1	1	1	0	6
x_5	0	0	3	1	1	1	10

Changing the Objective: Example

- The resultant tableau is not optimal:primal simplex
- The optimal tableau:

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	1	2	0	1	0	6
x_3	0	1	1	1	1	0	6
x_5	0	-1	2	0	0	1	4

- Since the "profit" realized on x_1 drops from 2 to zero, it is worth to reduce x_1 to zero in the solution and rather increase x_3 (substitute products/goods)
- Change in the RHS can be handled using duality
- Changing the RHS in the primal $=$ changing the objective in the dual $\Rightarrow$ perform sensitivity analysis on the dual

Parametric Analysis (Optional)

- In sensitivity analysis we ask how the optimum depends on certain model parameters: we change only a single parameter at a time
- **Parametric analysis:** what happens if more than one parameter changes along a known trajectory?
- Given the linear program $\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$, perturb the RHS vector $\boldsymbol{b}$ along a given direction $\boldsymbol{b}'$, while leaving the rest of the problem parameters intact:

$$\boldsymbol{b} + \lambda\boldsymbol{b}', \lambda \geq 0$$

- Parametric analysis can be used to characterize the optimal objective and the optimal solution for any $\lambda \geq 0$
- Again, we do not solve the problem from scratch
- See the full slide-deck for the details