Simplex: Starting Solution and Analysis

- Finding an initial basic feasible solution: the Artificial Variable technique
- Sensitivity analysis: the effect of changing the objective function
- Parametric analysis: perturbation of the Right-Hand-Side

Recall: The Simplex Tableau

- Let $\boldsymbol{A}$ be an $m \times n$ matrix with $\mathrm{rank}(\boldsymbol{A}) = \mathrm{rank}(\boldsymbol{A}, \boldsymbol{b}) = m$, $\boldsymbol{b}$ be a column m-vector, $\boldsymbol{x}$ be a column n-vector, and $\boldsymbol{c}^T$ be a row n-vector, and consider the linear program

$$\begin{aligned} z = \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- Let $\boldsymbol{B}$ be a primal feasible basis
- The simplex tableau

	z	$\boldsymbol{x_B}$	$\boldsymbol{x_N}$	RHS	
z	1	$\boldsymbol{0}$	$\boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c_N}^T$	$\boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{b}$	row 0
$\boldsymbol{x_B}$	$\boldsymbol{0}$	$\boldsymbol{I}_m$	$\boldsymbol{B}^{-1} \boldsymbol{N}$	$\boldsymbol{B}^{-1} \boldsymbol{b}$	rows 1..m

Recall: The Simplex Tableau

- Optimality condition: $\forall j \in N : z_j \geq 0$
- x_k enters the basis, where $k = \operatorname*{argmin}_{j \in N} z_j$
- Unbounded if no positive component in column k: $\boldsymbol{y}_k \leq 0$
- x_{B_r} leaves the basis: $r = \operatorname*{argmin}_{i \in \{1,\ldots,m\}} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$

	z	$x_{B_1} \ldots x_{B_r} \ldots x_{B_m}$	$\ldots x_{N_j} \ldots x_{N_k} \ldots$	RHS
z	1	$0 \;\ldots\; 0 \;\ldots\; 0$	$\ldots\; z_j \;\ldots\; z_k \;\ldots$	z_0
x_{B_1}	0	$1 \;\ldots\; 0 \;\ldots\; 0$	$\ldots\; y_{1j} \;\ldots\; y_{1k} \;\ldots$	$\bar{b}_1$
$\vdots$	$\vdots$	$\vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$
x_{B_r}	0	$0 \;\ldots\; 1 \;\ldots\; 0$	$\ldots\; y_{rj} \;\ldots\; y_{rk} \;\ldots$	$\bar{b}_r$
$\vdots$	$\vdots$	$\vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$
x_{B_m}	0	$0 \;\ldots\; 0 \;\ldots\; 1$	$\ldots\; y_{mj} \;\ldots\; y_{mk} \;\ldots$	$\bar{b}_m$

Recall: The Simplex Tableau

- We need an initial basic feasible solution to start the simplex
- In canonical form an initial basis is easy to find
- **Maximization problem:** $\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$
- Into standard form: $\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} + \boldsymbol{I}\boldsymbol{x}_s = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$
- If $\boldsymbol{b} \geq \boldsymbol{0}$ then the slack variables constitute a primal feasible initial basis: **primal simplex**
- For a **minimization problem** in canonical form:

$$\min\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$$

- Dual feasible initial basis on the columns of the slacks if $\boldsymbol{c}^T \geq \boldsymbol{0}^T$: **dual simplex**
- If neither case occurs then the simplex cannot be started: need a generic way for finding initial basic feasible solutions

Starting the Simplex Method

- Find an initial basic feasible solution for the linear program given in standard form:

$$\begin{aligned} z = \max \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

  where $\boldsymbol{A}$ is $m \times n$ with $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}, \boldsymbol{b}) = m$, $\boldsymbol{b}$ is a column m, $\boldsymbol{x}$ a column n, and $\boldsymbol{c}^T$ is a row n-vector
- Suppose furthermore that $\boldsymbol{b} \geq \boldsymbol{0}$ (if there is row i with $b_i < 0$, then invert the row to get $-b_i > 0$)
- If some columns of $\boldsymbol{A}$ form an identity matrix, choose these columns as basis
- Write the simplex tableau and eliminate reduced costs corresponding to the basis using elementary row operations

The Artificial Variable Technique

- Suppose that no trivial initial basis could be obtained this way
- Introduce $\boldsymbol{x_a}$ artificial variables and consider the modified linear program:

$$\begin{aligned} z = \min \quad & \mathbf{1}^T \boldsymbol{x_a} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{x_a} = \boldsymbol{b} \\ & \boldsymbol{x}, \boldsymbol{x_a} \geq \mathbf{0} \end{aligned}$$

  where $\mathbf{1}^T$ is a row vector (of proper size) with all components set to 1
- There is a trivial initial basis for the modified problem
- Since the columns of $\boldsymbol{x_a}$ form an identity matrix, we have a feasible initial basis on $\boldsymbol{x}_a$: $\boldsymbol{B} = \boldsymbol{I}$ and $\boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{b} \geq \mathbf{0}$ by assumption

The Artificial Variable Technique

- Solve the modified simplex from the initial basis defined by the artificial variables
- The optimum is $z_0 = \mathbf{1}^T \boldsymbol{x_a}$ (the sum of the artificial variables in the solution)
- **Thoerem:** if $z_0 > 0$ then the original linear program is infeasible
- **Proof:** suppose that $z_0 > 0$ but the original linear program is feasible so there is $\boldsymbol{x_0}$: $\boldsymbol{A x_0} = \boldsymbol{b}$
- Then, $\boldsymbol{x_a} = \boldsymbol{b} - \boldsymbol{A x_0} = \mathbf{0}$ and so $\mathbf{1}^T \boldsymbol{x_a} = 0 < z_0$, which contradicts the assumption that z_0 is optimal $\square$
- If, on the other hand, $z_0 = 0$, then $\boldsymbol{x_a} = \mathbf{0}$
- In this case the original linear program is feasible
- Solve it from the resultant basis

The Two–Phase Simplex Method

- **Phase One:** find an initial basis
- Solve the modified linear program augmented with the artificial variables $\boldsymbol{x_a}$

$$
\begin{aligned}
z = \max \quad & -\mathbf{1}^T \boldsymbol{x_a} \\
\text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} + \boldsymbol{x_a} = \boldsymbol{b} \\
& \boldsymbol{x}, \boldsymbol{x_a} \geq \mathbf{0}
\end{aligned}
$$

- If $\boldsymbol{x_a} \neq \mathbf{0}$ then the linear program is infeasible
- Otherwise, $\boldsymbol{x_a} = \mathbf{0}$ and suppose that all artificial variables have left the basis
- If not, the remaining artificial variables must be "pivoted" out from the basis manually, we do not discuss this here
- **Phase Two:** remove the artificial variables, restore the original objective function and run the simplex from the current basis

The Two–Phase Simplex: Example

- Solve the linear program using the Two–Phase Simplex

$$\begin{array}{lrcrcr} \min & -3x_1 & + & 4x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \leq & 4 \\ & 2x_1 & + & 3x_2 & \geq & 18 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

- Convert to maximization and bring to standard form by introducing slack variables (take note of the "$\geq$" type of constraints and that eventually we need $\boldsymbol{b} \geq \mathbf{0}$!)

$$\begin{array}{lrcrcrcrcr} \max & 3x_1 & - & 4x_2 & & & & & & \\ \text{s.t.} & x_1 & + & x_2 & + & x_3 & & & = & 4 \\ & 2x_1 & + & 3x_2 & & & - & x_4 & = & 18 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

The Two–Phase Simplex: Example

- No trivial primal or dual feasible basis
- Introduce artificial variables: it is enough add an artificial variable x_5 to the second row
- This, together with the slack variable x_3, will provide a proper initial (identity) basis
- Solve the below linear program as the first phase:

$$
\begin{array}{lccccccccccc}
\max & & & & & & & & & -x_5 & & \\
\text{s.t.} & x_1 & + & x_2 & + & x_3 & & & & & = & 4 \\
 & 2x_1 & + & 3x_2 & & & - & x_4 & + & x_5 & = & 18 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
\end{array}
$$

- Note that we have converted the objective to maximization: will need to invert the resultant objective value!

The Two–Phase Simplex: Example

- Initial basis: $\boldsymbol{B} = [\boldsymbol{a_3} \quad \boldsymbol{a_5}]$, $\boldsymbol{c_B}^T = [0 \quad -1]$, $\boldsymbol{c_N}^T = \boldsymbol{0}$
- Not a valid simplex tableau yet: there is a nonzero element in the objective row for the basic variable x_5
  - "pivot": subtract the row of x_5 from row 0

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	0	0	0	$\boxed{1}$	0
x_3	0	1	1	1	0	0	4
x_5	0	2	3	0	–1	$\boxed{1}$	18

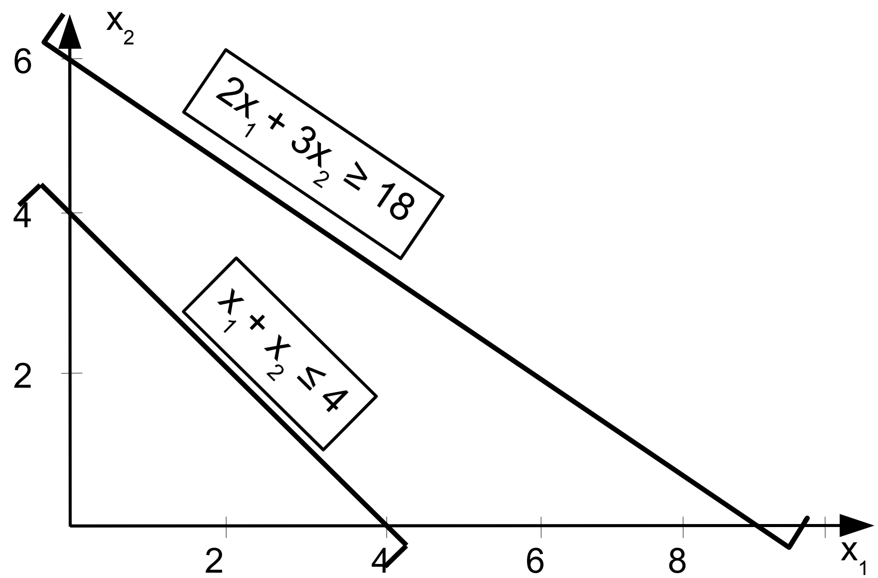
	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	–2	–3	0	1	0	–18
x_3	0	1	$\boxed{1}$	1	0	0	4
x_5	0	2	3	0	–1	1	18

The Two–Phase Simplex: Example

- We get an optimal tableau after the pivot, with optimal objective function value -6 (do not forget to invert this!)

	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	3	1	0	-6
x_2	1	1	1	0	0	4
x_5	-1	0	-3	-1	1	6

- Since $\min x_5 = 6$, the artificial variable could not be eliminated
- The original linear program is infeasible



The Two–Phase Simplex: Example

- Consider the linear program

$$\begin{array}{lrcrcr} \min & x_1 & - & 2x_2 & & \\ \text{s.t.} & -x_1 & - & x_2 & \leq & -2 \\ & -x_1 & + & x_2 & \geq & 1 \\ & & & x_2 & \leq & 3 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

- Invert the first two constraints so that $\boldsymbol{b} \geq \mathbf{0}$ holds
- Add an artificial variable to the first two rows (will use the slack variable for the third row)

$$\begin{array}{lrrrrrrrcr} \max & & & & & & -x_6 & -x_7 & & \\ \text{s.t.} & x_1 & +x_2 & -x_3 & & & +x_6 & & = & 2 \\ & -x_1 & +x_2 & & -x_4 & & & +x_7 & = & 1 \\ & & x_2 & & & +x_5 & & & = & 3 \\ & x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 & \geq & 0 \end{array}$$

The Two–Phase Simplex: Example

- Still need to take care of the objective row: 2 pivots

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	1	0	0	0	0	0	$\boxed{1}$	$\boxed{1}$	0
x_6	0	1	1	-1	0	0	$\boxed{1}$	0	2
x_7	0	-1	1	0	-1	0	0	$\boxed{1}$	1
x_5	0	0	1	0	0	1	0	0	3

	z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	1	0	-2	1	1	0	0	0	-3
x_6	0	1	1	-1	0	0	1	0	2
x_7	0	-1	$\boxed{1}$	0	-1	0	0	1	1
x_5	0	0	1	0	0	1	0	0	3

The Two–Phase Simplex: Example

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	-2	0	1	-1	0	0	2	-1
x_6	$\boxed{2}$	0	-1	1	0	1	-1	1
x_2	-1	1	0	-1	0	0	1	1
x_5	1	0	0	1	1	0	-1	2

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	RHS
z	0	0	0	0	0	1	1	0
x_1	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
x_2	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$
x_5	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{3}{2}$

The Two–Phase Simplex: Example

- Optimal tableau: end of Phase One
  - the objective function value is 0
  - the artificial variables have left the basis
  - so $x_6 = x_7 = 0$ can be removed
- The original objective: $\min x_1 - 2x_2 = -\max -x_1 + 2x_2$
- Signs change when written into the simplex tableau!
- Again not a valid tableau: 2 pivots

	x_1	x_2	x_3	x_4	x_5	RHS
z	$\boxed{1}$	$\boxed{-2}$	0	0	0	0
x_1	$\boxed{1}$	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$
x_2	0	$\boxed{1}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{3}{2}$
x_5	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$

The Two–Phase Simplex: Example

- Phase Two:

	x_1	x_2	x_3	x_4	x_5	RHS
z	0	0	$-\frac{1}{2}$	$-\frac{3}{2}$	0	$\frac{5}{2}$
x_1	1	0	$-\frac{1}{2}$	$\boxed{\frac{1}{2}}$	0	$\frac{1}{2}$
x_2	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{3}{2}$
x_5	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{3}{2}$

	x_1	x_2	x_3	x_4	x_5	RHS
z	3	0	-2	0	0	4
x_4	2	0	-1	1	0	1
x_2	1	1	-1	0	0	2
x_5	-1	0	$\boxed{1}$	0	1	1

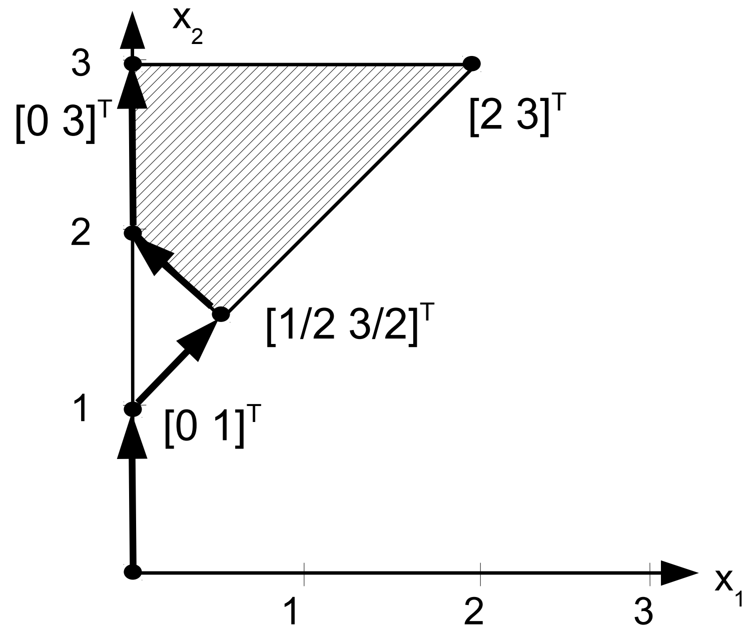
The Two–Phase Simplex: Example

	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	0	0	2	6
x_4	1	0	0	1	1	2
x_2	0	1	0	0	1	3
x_3	-1	0	1	0	1	1

- In Phase One, we have moved into an extreme point of the feasible region through the points

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} \frac{1}{2} \\ \frac{3}{2} \end{bmatrix}$$

- In Phase Two we have solved the problem from this point



Sensitivity Analysis

- Linear programs are often used to model real problems whose parameters are uncertain or subject to measurement errors or noise
- In a Resource Allocation problem, for instance, the estimated prices might be uncertain, capacities might be expanded by investing into new equipment, etc.
- **Question:** how does the optimal solution of a linear program $\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0}\}$ depend on the perturbation of the input parameters?
  - here we discuss only the case when the objective function coefficients $\boldsymbol{c}^T$ change
  - sensitivity analysis goes similarly for the cases when the RHS vector $\boldsymbol{b}$ or the constraint matrix $\boldsymbol{A}$ change
- The idea is that we do not want to re-optimize the changed linear program from scratch

Changing the Objective Function

- Let $\boldsymbol{B}$ be an optimal basis for the linear program $\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$
- We characterize the change in the optimal solution $\boldsymbol{x}$ and the optimal objective function value when the k-th objective function coefficient c_k is changed to c_k'
- The simplex tableau of the original linear program in the basis $\boldsymbol{B}$

	z	$\boldsymbol{x_B}$	$\boldsymbol{x_N}$	RHS	
z	1	$\boldsymbol{0}$	$\boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c_N}^T$	$\boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{b}$	row 0
$\boldsymbol{x_B}$	$\boldsymbol{0}$	$\boldsymbol{I}_m$	$\boldsymbol{B}^{-1}\boldsymbol{N}$	$\boldsymbol{B}^{-1}\boldsymbol{b}$	row 1...m

- $\boldsymbol{B}$ is (primal) feasible if $\boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$
- $\boldsymbol{B}$ is (primal) optimal if $\boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c_N}^T \geq \boldsymbol{0}$

Changing the Objective Function

1.) The changed objective coefficient c_k belongs to a nonbasic variable $x_k : k \in N$

$$\boldsymbol{c_N}^T \rightarrow (\boldsymbol{c'_N})^T = \boldsymbol{c_N} + (c'_k - c_k)\boldsymbol{e_k}^T$$

- In this case no change occurs in rows $1, \ldots, m$ of the simplex tableau, only the objective row (row 0) changes

$$\boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c_N}^T \rightarrow \boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} - (\boldsymbol{c_N}')^T =$$
$$\boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c_N}^T - (c'_k - c_k)\boldsymbol{e_k}^T$$

- In fact, only the reduced cost z_k for the nonbasic variable x_k changes:

$$z_k \rightarrow z'_k = z_k - (c'_k - c_k)$$

Changing the Objective Function

- If $z_k - (c'_k - c_k) \geq 0$ then basis $\boldsymbol{B}$ remains optimal
- For instance, if we **reduce** the cost of a nonbasic variable the current basis is guaranteed to remain optimal
- The objective function value does not change (x_k remains at 0)
- If, on the other hand, $z_k - (c'_k - c_k) < 0$, then basis $\boldsymbol{B}$ is no longer optimal according to the changed objective
- Run the primal simplex from basis $\boldsymbol{B}$ to obtain the new optimum
- Using this method we do not need to re-run the Two–Phase simplex from scratch, rather the simplex method continues from the optimal basis of the original problem
- This is the idea in sensitivity analysis

Changing the Objective Function

2.) The changed objective coefficient c_k belongs to a basic variable $x_k : k \in B$

- Let x_k be the t-th basic variable: $x_k \equiv x_{B_t}$

$$\boldsymbol{c_B}^T \to (\boldsymbol{c'_B})^T = \boldsymbol{c_B} + (c'_{B_t} - c_{B_t})\boldsymbol{e_t}^T$$

- Again, only the objective row changes in the tableau
- Basic variables (including x_{B_t}) still have zero reduced cost
- The reduced costs for nonbasic variables change, the j-th:

$$z'_j = (\boldsymbol{c'_B})^T \boldsymbol{B}^{-1} \boldsymbol{a}_j - c_j = \boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{a}_j - c_j +$$
$$[0 \quad 0 \ \dots \ c'_{B_t} - c_{B_t} \ \dots \quad 0]\boldsymbol{y_j} = z_j + (c'_{B_t} - c_{B_t}) y_{tj}$$

- Add $c'_{B_t} - c_{B_t}$ times the row of x_{B_t} to row 0
- Then zero out the reduced cost for x_{B_t}

Changing the Objective: Example

- Solve the below linear program:

$$\begin{array}{lrcrcrcl} \max & 2x_1 & - & x_2 & + & x_3 & & \\ \text{s.t.} & x_1 & + & x_2 & + & x_3 & \leq & 6 \\ & -x_1 & + & 2x_2 & & & \leq & 4 \\ & x_1, & & x_2, & & x_3 & \geq & 0 \end{array}$$

- The slack variables form a feasible initial basis

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	-2	1	-1	0	0	0
x_4	0	$\boxed{1}$	1	1	1	0	6
x_5	0	-1	2	0	0	1	4

Changing the Objective: Example

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	3	1	2	0	12
x_1	0	1	1	1	1	0	6
x_5	0	0	3	1	1	1	10

- Optimal tableau, with basic variables $B = \{1, 5\}$
- Reduce $c_2 = -1$ to $c'_2 = -3$: since x_2 is not basic only the reduced cost z_2 changes in row 0:
  $z'_2 = z_2 - (c'_2 - c_2) = 3 - (-3 - (-1)) = 5$

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	5	1	2	0	12
...	...	...	...	...	...	...	...

- The tableau remains optimal, the objective function value does not change

Changing the Objective: Example

- If now the objective coefficient for x_2 is changed to $c_2' = 3$, then $z_2' = -1$
- The resultant tableau is no longer optimal: primal simplex

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	-1	1	2	0	12
x_1	0	1	1	1	1	0	6
x_5	0	0	$\boxed{3}$	1	1	1	10

- The optimal tableau

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	0	$\frac{4}{3}$	$\frac{7}{3}$	$\frac{1}{3}$	$\frac{46}{3}$
x_1	0	1	0	$\frac{2}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{8}{3}$
x_2	0	0	1	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{10}{3}$

Changing the Objective: Example

- Now change the cost for a basic variable, say, x_1, from $c_1 = 2$ to zero
- The optimal tableau of the original problem:

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	3	1	2	0	12
x_1	0	1	1	1	1	0	6
x_5	0	0	3	1	1	1	10

- Add the first row to row 0 exactly $c_1' - c_1 = -2$ times (that is, subtract the double)

Changing the Objective: Example

- Performing the row operation, the objective value changes:

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	$\boxed{-2}$	1	-1	0	0	0
x_1	0	1	1	1	1	0	6
x_5	0	0	3	1	1	1	10

- Since only the elements that belong to nonbasic variables need to be altered in the objective row, we simple set the reduced cost for x_1 to zero

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	1	-1	0	0	0
x_1	0	1	1	$\boxed{1}$	1	0	6
x_5	0	0	3	1	1	1	10

Changing the Objective: Example

- The resultant tableau is not optimal:primal simplex
- The optimal tableau:

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	1	2	0	1	0	6
x_3	0	1	1	1	1	0	6
x_5	0	-1	2	0	0	1	4

- Since the “profit” realized on x_1 drops from 2 to zero, it is worth to reduce x_1 to zero in the solution and rather increase x_3 (substitute products/goods)
- Change in the RHS can be handled using duality
- Changing the RHS in the primal $=$ changing the objective in the dual $\Rightarrow$ perform sensitivity analysis on the dual

Parametric Analysis

- In sensitivity analysis we ask how the optimum depends on certain model parameters
- We change only a single parameter at a time
- In practice the question is often arises what happens if more than one parameters change according to some given perturbation function
- For instance, in the optimal product mix purchase prices might change simultaneously
- **Parametric analysis**
  - we now discuss only the case when the RHS is perturbed along a given direction
  - parametric analysis on the objective can again be traced back to parametric analysis on the RHS of the dual

Perturbation of the RHS

- Consider the linear program

$$\begin{aligned} z = \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

  where $\boldsymbol{A}$ is $m \times n$ with $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}, \boldsymbol{b}) = m$, $\boldsymbol{b}$ is a column m, $\boldsymbol{x}$ a column n, and $\boldsymbol{c}^T$ is a row n-vector

- Perturb the RHS vector $\boldsymbol{b}$ along a given direction $\boldsymbol{b}'$, while leaving the rest of the problem parameters intact:

$$\boldsymbol{b} + \lambda \boldsymbol{b}', \lambda \geq 0$$

- Characterize the change in the optimal objective and in the optimal solution for $\lambda \geq 0$

Perturbation of the RHS

- Consider the simplex tableau for a basis $\boldsymbol{B}$ that is optimal for the unchanged problem

	z	$\boldsymbol{x_B}$	$\boldsymbol{x_N}$	RHS	
z	1	$\mathbf{0}$	$\boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c_N}^T$	$\boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{b}$	row 0
$\boldsymbol{x_B}$	$\mathbf{0}$	$\boldsymbol{I}_m$	$\boldsymbol{B}^{-1}\boldsymbol{N}$	$\boldsymbol{B}^{-1}\boldsymbol{b}$	rows 1...m

- Since the tableau us optimal: $\boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c_N}^T \geq \mathbf{0}$
- This does not depend on $\boldsymbol{b}$ so the tableau remains primal optimal for any λ
- The question is how long it remains primal feasible as well?
- Perturbing $\boldsymbol{b}$ effects only the RHS column by $\boldsymbol{B}^{-1}(\boldsymbol{b} + \lambda\boldsymbol{b}')$ and the objective value by $\boldsymbol{c_B}^T\boldsymbol{B}^{-1}(\boldsymbol{b} + \lambda\boldsymbol{b}')$
- The tableau remains primal feasible as long as the RHS is nonnegative: $\boldsymbol{B}^{-1}(\boldsymbol{b} + \lambda\boldsymbol{b}') = \boldsymbol{B}^{-1}\boldsymbol{b} + \lambda(\boldsymbol{B}^{-1}\boldsymbol{b}') \geq \mathbf{0}$

Perturbation of the RHS

- Let $S = \{i : \bar{b}'_i < 0\}$, where $\bar{b}'_i$ is the i-th component of $\bar{\boldsymbol{b}}' = \boldsymbol{B}^{-1}\boldsymbol{b}'$
- If $S = \emptyset$ then $\boldsymbol{B}^{-1}(\boldsymbol{b} + \lambda\boldsymbol{b}') \geq \mathbf{0}$ for any $\lambda \geq 0$ and thus basis $\boldsymbol{B}$ remains optimal unconditionally
- Otherwise, find the row i for which $\bar{b}_i + \lambda\bar{b}'_i$ first becomes negative:

$$r = \operatorname*{argmin}_{i \in S} \left(-\frac{\bar{b}_i}{\bar{b}'_i} \right), \qquad \bar{\lambda} = \min_{i \in S} \left(-\frac{\bar{b}_i}{\bar{b}'_i} \right) = -\frac{\bar{b}_r}{\bar{b}'_r}$$

- For any $0 \leq \lambda \leq \bar{\lambda} = \lambda_1$ basis $\boldsymbol{B}$ is feasible and optimal
- Meanwhile, the objective function value and the variables:

$$z(\lambda) = \boldsymbol{c_B}^T(\bar{\boldsymbol{b}} + \lambda\bar{\boldsymbol{b}}'), \qquad \boldsymbol{x}(\lambda) = \begin{bmatrix} \boldsymbol{x_B} \\ \boldsymbol{x_N} \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{b}} + \lambda\bar{\boldsymbol{b}}' \\ \mathbf{0} \end{bmatrix}$$

Perturbation of the RHS

- On the other hand, for any $\lambda > \bar{\lambda}$ basis $\boldsymbol{B}$ is no longer primal feasible
- Dual simplex pivot on x_{B_r}
- If there is no blocking variable (i.e., when the row of x_{B_r} is nonnegative), then the primal problem becomes infeasible (dual unbounded) for any $\lambda > \bar{\lambda}$
- Otherwise, we find the parameter $\bar{\lambda} = \lambda_2$ until which the tableau obtained remains primal feasible
- We carry on with this iteration until we get either
  - $S = \emptyset$: in this case the current basis is optimal form any $\lambda > \bar{\lambda}$, or
  - no blocking variable is found during the dual simplex pivot: the primal is infeasible for any $\lambda > \bar{\lambda}$

Perturbation of the RHS: Example

- Solve the below linear program, subject to the following perturbation of the RHS:

$$\begin{array}{lrcrcl} \max & x_1 & + & 3x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \leq & 6-\lambda \\ & -x_1 & + & 2x_2 & \leq & 6+\lambda \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

- Easily, $\boldsymbol{b} = [6 \quad 6]^T$ and $\boldsymbol{b'} = [-1 \quad 1]^T$
- Adding slacks x_3 and x_4 the optimal solution for $\lambda = 0$:

	z	x_1	x_2	x_3	x_4	RHS
z	1	0	0	$\frac{5}{3}$	$\frac{2}{3}$	14
x_1	0	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	2
x_2	0	0	1	$\frac{1}{3}$	$\frac{1}{3}$	4

Perturbation of the RHS: Example

- Question: how long the current basis $\boldsymbol{B} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$ remains optimal?
- Observe that the columns of the slack variables x_3 and x_4 in the original problem form an identity matrix
- Therefore the corresponding columns in the tableau specify precisely the inverse of $\boldsymbol{B}$:

$$\bar{\boldsymbol{b}}' = \boldsymbol{B}^{-1}\boldsymbol{b}' = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

- From this: $S = \{1\}$, $r = 1$, and $\bar{\lambda} = -\frac{\bar{b}_1}{\bar{b}'_1} = -\frac{2}{-1} = 2 = \lambda_1$

Perturbation of the RHS: Example

- Consequently, for $\lambda \in [0,1]$ the optimal objective function value and the optimal solution:

$$z(\lambda) = \boldsymbol{c_B}^T(\bar{\boldsymbol{b}} + \lambda\bar{\boldsymbol{b}}') = [1 \quad 3]\begin{bmatrix}2\\4\end{bmatrix} + \lambda[1 \quad 3]\begin{bmatrix}-1\\0\end{bmatrix} = 14 - \lambda$$

$$\boldsymbol{x}(\lambda) = \begin{bmatrix}x_1(\lambda)\\x_2(\lambda)\end{bmatrix} = \bar{\boldsymbol{b}} + \lambda\bar{\boldsymbol{b}}' = \begin{bmatrix}2\\4\end{bmatrix} + \lambda\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}2-\lambda\\4\end{bmatrix}$$

- For $\lambda > 2$ we have $x_1 < 0$ in the current basis

	z	x_1	x_2	x_3	x_4	RHS
z	1	0	0	$\frac{5}{3}$	$\frac{2}{3}$	$14-\lambda$
x_1	0	1	0	$\frac{2}{3}$	$\boxed{-\frac{1}{3}}$	$2-\lambda$
x_2	0	0	1	$\frac{1}{3}$	$\frac{1}{3}$	4

Perturbation of the RHS: Example

- Choose $\lambda = 2$ and perform a dul simplex pivot on the basic variable x_1: x_4 enters the basis

	z	x_1	x_2	x_3	x_4	RHS
z	1	2	0	3	0	12
x_4	0	-3	0	-2	1	0
x_2	0	1	1	1	0	4

- Now $\bar{\boldsymbol{b}} = \begin{bmatrix} \bar{b}_4 \\ \bar{b}_2 \end{bmatrix}$ and we search for $\bar{\boldsymbol{b}}' = \begin{bmatrix} \bar{b}'_4 \\ \bar{b}'_2 \end{bmatrix}$ according to $\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$ and $\boldsymbol{b}' = \begin{bmatrix} b'_1 \\ b'_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$
- Multiplying by the matrix formed by column 3 and 4:

$$\bar{\boldsymbol{b}} = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} -6 \\ 6 \end{bmatrix} \quad \bar{\boldsymbol{b}}' = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Perturbation of the RHS: Example

- Then, $S = \{2\}$, $r = 2$, and $\bar{\lambda} = -\frac{\bar{b}_2}{\bar{b}'_2} = -\frac{6}{-1} = 6 = \lambda_2$
- The objective function value and the solution for $\lambda \in [2, 6]$:

$$z(\lambda) = \boldsymbol{c_B}^T(\bar{\boldsymbol{b}} + \lambda\bar{\boldsymbol{b}}') = [0 \quad 3]\begin{bmatrix} -6 \\ 6 \end{bmatrix} + \lambda[0 \quad 3]\begin{bmatrix} 3 \\ -1 \end{bmatrix} = 18 - 3\lambda$$

$$x_1(\lambda) \equiv 0, \qquad \begin{bmatrix} x_4(\lambda) \\ x_2(\lambda) \end{bmatrix} = \bar{\boldsymbol{b}} + \lambda\bar{\boldsymbol{b}}' = \begin{bmatrix} -6 \\ 6 \end{bmatrix} + \lambda\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -6 + 3\lambda \\ 6 - \lambda \end{bmatrix}$$

	z	x_1	x_2	x_3	x_4	RHS
z	1	2	0	3	0	$18 - 3\lambda$
x_4	0	-3	0	-2	1	$-6 + 3\lambda$
x_2	0	1	1	1	0	$6 - \lambda$

Perturbation of the RHS: Example

- The current basis is no longer feasible for any $\lambda > 6$
- Move the a new basis: x_2 leaves the basis
- Since the row of x_2 is nonnegative, we have dual unboundedness at this point
- Consequently, for any $\lambda > 6$ the primal becomes infeasible

- The objective function value is **piecewise linear** and **concave**