The Dual Simplex Algorithm: A Summary

WARNING: this is just a summary of the material covered in the full slide-deck ***The Dual Simplex Algorithm*** *that will orient you as per the topics covered there; you are required to learn the full version, not just this summary!*

- Primal optimal (dual feasible) and primal feasible (dual optimal) bases
- The dual simplex tableau, dual optimality and the dual pivot rules
- Lots of exercises (in the slide-deck)

Recall: Linear Programming Duality

- Consider the (primal) maximization (linear) program:

$$\begin{aligned} z = \max \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

  where $\boldsymbol{A}$ is an $m \times n$ matrix, $\boldsymbol{b}$ is a column m-vector, $\boldsymbol{x}$ is a column n-vector, and $\boldsymbol{c}^T$ is a row n-vector

- The dual is a standard form minimization problem, where $\boldsymbol{w}^T$ is a row m-vector and $\boldsymbol{v}^T$ is a row n-vector

$$\begin{aligned} & \min \boldsymbol{w}^T\boldsymbol{b} \\ & \text{s.t. } \boldsymbol{w}^T\boldsymbol{A} - \boldsymbol{v}^T = \boldsymbol{c}^T \\ & \qquad \boldsymbol{v}^T \geq \boldsymbol{0}, \ \boldsymbol{w}^T \text{ arbitrary} \end{aligned}$$

Strong Duality: Another Approach

- Let $\boldsymbol{B}$ be an *any* basis, let $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x_B} \\ \boldsymbol{x_N} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}$ be the corresponding solution in the primal, and choose the dual variables as follows:

$$\boldsymbol{w}^T = \boldsymbol{c_B}^T \boldsymbol{B}^{-1}, \qquad \boldsymbol{v}^T = [\underbrace{\boldsymbol{0}}_{\text{basic}} \quad \underbrace{\boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c_N}^T}_{\text{nonbasic}}]$$

- We can show that this choice satisfies the dual conditions at least partially
- First, the objective function value of the dual **identically** equals the primal objective function value in the basis $\boldsymbol{B}$:

$$\boldsymbol{w}^T \boldsymbol{b} = \boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{b} = \boldsymbol{c_B}^T \boldsymbol{x_B} = \boldsymbol{c}^T \boldsymbol{x}$$

Strong Duality: Another Approach

- Second, the dual condition $\boldsymbol{w}^T \boldsymbol{A} - \boldsymbol{v}^T = \boldsymbol{c}^T$ also holds with identity:

$$\begin{aligned}\boldsymbol{w}^T \boldsymbol{A} - \boldsymbol{v}^T &= \boldsymbol{c_B}^T \boldsymbol{B}^{-1} [\boldsymbol{B} \quad \boldsymbol{N}] - \left[\boldsymbol{0} \quad (\boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c_N}^T)\right] \\ &= [\boldsymbol{c_B}^T \quad (\boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} + \boldsymbol{c_N}^T) = \boldsymbol{c}^T\end{aligned}$$

- The only dual condition that may not hold is $\boldsymbol{v}^T \geq \boldsymbol{0}$
- It is easy to show that this condition holds if and only if $\boldsymbol{B}$ is an optimal basis
- This is because in $\boldsymbol{v}^T = \left[\boldsymbol{0} \quad (\boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c_N}^T)\right]$ the component $\boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c_N}^T$ coincides with row 0 of the optimal simplex table
- But this must is non-negative for an optimal simplex table, so we have $\boldsymbol{v}^T \geq \boldsymbol{0}$ as required

The Dual Simplex Method

- We could say that **the primal simplex method develops a basis that satisfies all the dual conditions simultaneously**
- In each iteration it uniquely determines the value of the dual variables $\boldsymbol{w}^T$ and $\boldsymbol{v}^T$ so that
  - the dual objective function value equals the primal objective function value: $\boldsymbol{w}^T\boldsymbol{b} \equiv \boldsymbol{c}^T\boldsymbol{x}$
  - the dual condition $\boldsymbol{w}^T\boldsymbol{A} - \boldsymbol{v}^T \equiv \boldsymbol{c}^T$ identically holds
- In addition, at optimality it also satisfies the dual non-negativity conditions $\boldsymbol{v}^T \geq \mathbf{0}$
- The **dual simplex method** is the "dual" of the primal simplex: it converges through a series of "dual feasible" bases into a "dual optimal" (primal feasible) basis
  - in every iteration it fulfills (D), (CS) and (P) partially
  - optimality when (P) is fully satisfied

The Dual Simplex Method

- Consider the standard form linear program:

$$\begin{aligned} \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

  where $\boldsymbol{A}$ is an $m \times n$ matrix, $\boldsymbol{b}$ is a column m-vector, $\boldsymbol{x}$ is a column n-vector, and $\boldsymbol{c}^T$ is a row n-vector

- Let $\boldsymbol{B}$ be a basis that satisfies
  - the primal optimality conditions (i.e., **dual feasible**)

$$\boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c_N}^T \geq \boldsymbol{0}$$

  - but is not primal feasible (i.e., not **dual optimal**)

$$\boldsymbol{B}^{-1} \boldsymbol{b} \not\geq \boldsymbol{0}$$

The Dual Simplex Method

- The simplex tableau for basis $\boldsymbol{B}$
  - (dual) feasible if $\forall j \in N : z_j \geq 0$
  - (dual) optimal, if $\forall i \in \{1, \ldots, m\} : \bar{b}_i \geq 0$
- The goal is to obtain a simplex tableau that is dual optimal, maintaining dual feasibility along the way
- In terms of the tableau, this means that
  - in row 0 we always have nonnegative elements (dual feasibility)
  - but the RHS column may contain negative elements (not dual optimal)
- Eventually, the RHS column will also become nonnegative
- This is attained through a sequence of (dual) pivots
- For brevity, we merely state the method without proofs

The Dual Simplex Method

- Choose the **leaving variable** x_r first as the basic variable with the smallest value in the current basis:

$$r = \operatorname*{argmin}_{i \in \{1,\ldots,m\}} \bar{b}_i$$

- **Lemma:** after the pivot we obtain a primal optimal basic feasible solution (row 0 is nonnegative), if the **entering variable** x_k is chosen according to:

$$k = \operatorname*{argmin}_{j \in N} \left\{ -\frac{z_j}{y_{rj}} : y_{rj} < 0 \right\}$$

- **Lemma:** if $\forall j \in N : y_{rj} \geq 0$, then the dual is unbounded and the primal is infeasible
- Pivot on row r and column k

The Dual Simplex Method: Example

- Consider the linear program

$$
\begin{array}{lcccccccc}
\min & 2x_1 & + & 3x_2 & + & 4x_3 & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & x_3 & \geq & 3 \\
 & 2x_1 & - & x_2 & + & 3x_3 & \geq & 4 \\
 & x_1, & & x_2, & & x_3 & \geq & 0
\end{array}
$$

- Bringing to standard form and converting to maximization (note the eventual inversion!):

$$
\begin{array}{lcccccccccc}
\max & -2x_1 & - & 3x_2 & - & 4x_3 & & & & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & x_3 & - & x_4 & & & = & 3 \\
 & 2x_1 & - & x_2 & + & 3x_3 & & & - & x_5 & = & 4 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
\end{array}
$$

- Cannot use the primal simplex since the initial basis formed by the slack variables is not (primal) feasible

The Dual Simplex Method: Example

- Let us use the dual simplex (after inverting the constraints):

$$\begin{array}{lrcrcrcrcrcr}
\max & -2x_1 & - & 3x_2 & - & 4x_3 & & & & & & \\
\text{s.t.} & -x_1 & - & 2x_2 & - & x_3 & + & x_4 & & & = & -3 \\
 & -2x_1 & + & x_2 & - & 3x_3 & & & + & x_5 & = & -4 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0
\end{array}$$

- We can do this since the slack variables for a primal optimal (dual feasible) initial basis

$$\boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c_N}^T = \mathbf{0} - [-2 \ \ -3 \ \ -4] \geq \mathbf{0}$$

- Not dual optimal: $\boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{I}_2 \begin{bmatrix} -3 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix} \ngeq \mathbf{0}$

The Dual Simplex Method: Example

- The initial simplex tableau:

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	2	3	4	0	0	0
x_4	0	-1	-2	-1	1	0	-3
x_5	0	$\boxed{-2}$	1	-3	0	1	-4

- The most negative basic variable leaves the basis: x_5
- The entering variable is x_1 as $-\frac{z_1}{y_{51}} = \min\{-\frac{z_j}{y_{5j}} : y_{5j} < 0\}$
- Divide the j-the element of row 0 with the j-th element of the r-th row if that is negative and invert, and take the minimum
- If we choose the leaving and entering variable this way, we get a dual feasible basis after the pivot

The Dual Simplex Method: Example

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	2	3	4	0	0	0
x_4	0	-1	-2	-1	1	0	-3
x_1	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	$-\frac{1}{2}$	2

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	2	3	4	0	0	0
x_4	0	0	$-\frac{5}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	-1
x_1	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	$-\frac{1}{2}$	2

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	4	1	0	1	-4
x_4	0	0	$\boxed{-\frac{5}{2}}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	-1
x_1	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	$-\frac{1}{2}$	2

The Dual Simplex Method: Example

- The new basis is dual feasible (primal optimal) but still not dual optimal, as $x_4 = -1 < 0$
- x_4 leaves the basis and $x_2 = \text{argmin}\{-\frac{z_j}{y_{4j}} : y_{4j} < 0\}$ enters

	x_1	x_2	x_3	x_4	x_5	RHS
z	0	0	$\frac{9}{5}$	$\frac{8}{5}$	$\frac{1}{5}$	$-\frac{28}{5}$
x_2	0	1	$-\frac{1}{5}$	$-\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
x_1	1	0	$\frac{7}{5}$	$-\frac{1}{5}$	$-\frac{2}{5}$	$\frac{11}{5}$

- The resultant basis is both dual optimal and dual feasible
- The optimum for the maximization problem is $z = -\frac{28}{5}$, attained at the point $\boldsymbol{x} = [\frac{11}{5} \quad \frac{2}{5} \quad 0]^T$

The Dual Simplex Method: Example

- Observe that the objective function value for the maximization problem has decreased in each iteration

$$\max -2x_1 - 3x_2 - 4x_3 : 0 \to -4 \to -\frac{28}{5}$$

- Of course, this is because we have in fact solved the dual minimization problem $\min\{\boldsymbol{w}^T\boldsymbol{b} : \boldsymbol{w}^T\boldsymbol{A} \geq \boldsymbol{c}^T\}$
- Choosing $\boldsymbol{w}^T = \boldsymbol{c_B}\boldsymbol{B}^{-1}$ the dual objective function $\boldsymbol{w}^T\boldsymbol{b} = \boldsymbol{c_B}\boldsymbol{B}^{-1}\boldsymbol{b}$ can be read from the simplex tableau in each step (row zero, RHS column)

$$\min \boldsymbol{w}^T\boldsymbol{b} : 0 \to -4 \to -\frac{28}{5}$$

- Originally we had a minimization problem (invert!), whose optimum is thus $z = \frac{28}{5}$