The Dual Simplex Algorithm

- Primal optimal (dual feasible) and primal feasible (dual optimal) bases
- The dual simplex tableau, dual optimality and the dual pivot rules
- Classical applications of linear programming: the use of the primal and the dual simplex methods, examples

Recall: Linear Programming Duality

- Consider the (primal) linear program:

$$\begin{aligned} z = \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

  where $\boldsymbol{A}$ is an $m \times n$ matrix, $\boldsymbol{b}$ is a column m-vector, $\boldsymbol{x}$ is a column n-vector, and $\boldsymbol{c}^T$ is a row n-vector

- By the Karush-Kuhn-Tucker Conditions, $\boldsymbol{x}$ is optimal if and only if there is $(\boldsymbol{v}^T, \boldsymbol{w}^T)$ so that

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \qquad \boldsymbol{x} \geq \boldsymbol{0} \tag{P}$$

$$\boldsymbol{c}^T - \boldsymbol{w}^T\boldsymbol{A} + \boldsymbol{v}^T = \boldsymbol{0}, \qquad \boldsymbol{v}^T \geq \boldsymbol{0} \tag{D}$$

$$\boldsymbol{v}^T\boldsymbol{x} = 0 \tag{CS}$$

Recall: Linear Programming Duality

- Let $\boldsymbol{x}$ be a basic feasible solution and let $\boldsymbol{B}$ denote the corresponding basis matrix

	z	$\boldsymbol{x_B}$	$\boldsymbol{x_N}$	RHS	
z	1	$\boldsymbol{0}$	$\boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c_N}^T$	$\boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{b}$	0. sor
$\boldsymbol{x_B}$	$\boldsymbol{0}$	$\boldsymbol{I}_m$	$\boldsymbol{B}^{-1}\boldsymbol{N}$	$\boldsymbol{B}^{-1}\boldsymbol{b}$	1..m sorok

- Choose the dual variables as follows:

$$\boldsymbol{w}^T = \boldsymbol{c_B}^T\boldsymbol{B}^{-1}, \qquad \boldsymbol{v}^T = [\underbrace{\boldsymbol{0}}_{\text{basic}} \quad \underbrace{\boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c_N}^T}_{\text{nonbasic}}]$$

- (P) holds since $\boldsymbol{x}$ is feasible
- (CS) holds identically since

$$\boldsymbol{v}^T\boldsymbol{x} = \boldsymbol{0}\boldsymbol{x_B} + (\boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c_N}^T)\boldsymbol{0} \equiv 0$$

Recall: Linear Programming Duality

- One of the constraints of (D), namely $\boldsymbol{c}^T - \boldsymbol{w}^T\boldsymbol{A} + \boldsymbol{v}^T = \boldsymbol{0}$ also holds identically
- Separating to basic and nonbasic components:

$$\boldsymbol{c}^T - \boldsymbol{w}^T\boldsymbol{A} + \boldsymbol{v}^T = ({\boldsymbol{c_B}}^T, {\boldsymbol{c_N}}^T) - \boldsymbol{w}^T(\boldsymbol{B}, \boldsymbol{N}) + (\boldsymbol{0}, {\boldsymbol{c_B}}^T\boldsymbol{B}^{-1}\boldsymbol{N} - {\boldsymbol{c_N}}^T)$$

- Component-wise:

$${\boldsymbol{c_B}}^T - \boldsymbol{w}^T\boldsymbol{B} + \boldsymbol{0} = {\boldsymbol{c_B}}^T - {\boldsymbol{c_B}}^T\boldsymbol{B}^{-1}\boldsymbol{B} \equiv \boldsymbol{0} \qquad \text{(basic)}$$

$${\boldsymbol{c_N}}^T - \boldsymbol{w}^T\boldsymbol{N} + ({\boldsymbol{c_B}}^T\boldsymbol{B}^{-1}\boldsymbol{N} - {\boldsymbol{c_N}}^T) = -{\boldsymbol{c_B}}^T\boldsymbol{B}^{-1}\boldsymbol{N} + {\boldsymbol{c_B}}^T\boldsymbol{B}^{-1}\boldsymbol{N} \equiv \boldsymbol{0} \qquad \text{(nonbasic)}$$

- The other part of (D), $\boldsymbol{v}^T \geq \boldsymbol{0}$, only holds if $\boldsymbol{B}$ is optimal

The Dual Simplex Method

- Correspondingly, the primal simplex method develops a basis that satisfies the (P), (D), and (CS) conditions simultaneously
- In each iteration it satisfies the primal conditions (P), the complementary slackness conditions (CS), and the dual conditions (D) partially
- We have optimality when (D) is fully satisfied
- The **dual simplex method** is the “dual” of the primal simplex: it converges through a series of “dual feasible” bases into a “dual optimal” (primal feasible) basis
  - in every iteration it fulfills (D), (CS) and (P) partially
  - optimality when (P) is fully satisfied
- Useful when it is easy to find a dual feasible (primal optimal) initial basis

The Dual Simplex Method

- Consider the standard form linear program:

$$\begin{aligned} \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

  where $\boldsymbol{A}$ is an $m \times n$ matrix, $\boldsymbol{b}$ is a column m-vector, $\boldsymbol{x}$ is a column n-vector, and $\boldsymbol{c}^T$ is a row n-vector

- Let $\boldsymbol{B}$ be a basis that satisfies
  - the primal optimality conditions (i.e., **dual feasible**)

$$\boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c_N}^T \geq \boldsymbol{0}$$

  - but is not primal feasible (i.e., not **dual optimal**)

$$\boldsymbol{B}^{-1} \boldsymbol{b} \ngeq \boldsymbol{0}$$

The Dual Simplex Method

- The simplex tableau for basis $\boldsymbol{B}$
  - (dual) feasible if $\forall j \in N : z_j \geq 0$
  - (dual) optimal, if $\forall i \in \{1, \ldots, m\} : \bar{b}_i \geq 0$
- The goal is to obtain a simplex tableau that is dual optimal, maintaining dual feasibility along the way
- In terms of the tableau, this means that
  - in row 0 we always have nonnegative elements (dual feasibility)
  - but the RHS column may contain negative elements (not dual optimal)
- Eventually, the RHS column will also become nonnegative
- This is attained through a sequence of (dual) pivots
- For brevity, we merely state the method without proofs

The Dual Simplex Method

- Choose the **leaving variable** x_r first as the basic variable with the smallest value in the current basis:

$$r = \operatorname*{argmin}_{i \in \{1,\ldots,m\}} \bar{b}_i$$

- **Lemma:** after the pivot we obtain a primal optimal basic feasible solution (row 0 is nonnegative), if the **entering variable** x_k is chosen according to:

$$k = \operatorname*{argmin}_{j \in N} \left\{ -\frac{z_j}{y_{rj}} : y_{rj} < 0 \right\}$$

- **Lemma:** if $\forall j \in N : y_{rj} \geq 0$, then the dual is unbounded and the primal is infeasible
- Pivot on row r and column k

The Dual Simplex Method: Example

- Consider the linear program

$$\begin{array}{lrcrcrcl} \min & 2x_1 & + & 3x_2 & + & 4x_3 & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & x_3 & \geq & 3 \\ & 2x_1 & - & x_2 & + & 3x_3 & \geq & 4 \\ & x_1, & & x_2, & & x_3 & \geq & 0 \end{array}$$

- Bringing to standard form and converting to maximization (note the eventual inversion!):

$$\begin{array}{lrcrcrcrcrcl} \max & -2x_1 & - & 3x_2 & - & 4x_3 & & & & & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & x_3 & - & x_4 & & & = & 3 \\ & 2x_1 & - & x_2 & + & 3x_3 & & & - & x_5 & = & 4 \\ & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0 \end{array}$$

- Cannot use the primal simplex since the initial basis formed by the slack variables is not (primal) feasible

The Dual Simplex Method: Example

- Let us use the dual simplex (after inverting the constraints):

$$\begin{array}{rrcrcrcrcrcr} \max & -2x_1 & - & 3x_2 & - & 4x_3 & & & & & & \\ \text{s.t.} & -x_1 & - & 2x_2 & - & x_3 & + & x_4 & & & = & -3 \\ & -2x_1 & + & x_2 & - & 3x_3 & & & + & x_5 & = & -4 \\ & x_1, & & x_2, & & x_3, & & x_4, & & x_5 & \geq & 0 \end{array}$$

- We can do this since the slack variables for a primal optimal (dual feasible) initial basis

$$\boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c_N}^T = \boldsymbol{0} - [-2 \; -3 \; -4] \geq \boldsymbol{0}$$

- Not dual optimal: $\boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{I}_2 \begin{bmatrix} -3 \\ -4 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \end{bmatrix} \not\geq \boldsymbol{0}$

The Dual Simplex Method: Example

- The initial simplex tableau:

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	2	3	4	0	0	0
x_4	0	-1	-2	-1	1	0	-3
x_5	0	$\boxed{-2}$	1	-3	0	1	-4

- The most negative basic variable leaves the basis: x_5
- The entering variable is x_1 as $-\frac{z_1}{y_{51}} = \min\{-\frac{z_j}{y_{5j}} : y_{5j} < 0\}$
- Divide the j-the element of row 0 with the j-th element of the r-th row if that is negative and invert, and take the minimum
- If we choose the leaving and entering variable this way, we get a dual feasible basis after the pivot

The Dual Simplex Method: Example

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	2	3	4	0	0	0
x_4	0	-1	-2	-1	1	0	-3
x_1	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	$-\frac{1}{2}$	2

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	2	3	4	0	0	0
x_4	0	0	$-\frac{5}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	-1
x_1	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	$-\frac{1}{2}$	2

	z	x_1	x_2	x_3	x_4	x_5	RHS
z	1	0	4	1	0	1	-4
x_4	0	0	$\boxed{-\frac{5}{2}}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	-1
x_1	0	1	$-\frac{1}{2}$	$\frac{3}{2}$	0	$-\frac{1}{2}$	2

The Dual Simplex Method

- After the pivot the RHS element of the pivot row is always nonnegative, since first we divided the row of x_r by $y_{rk} < 0$ and so we invert all elements, this way $\bar{b}_r < 0$ as well
- If the basis is not dual degenerate ($z_k > 0$), then after the pivot the objective function value **decreases**
- In fact, the current basis satisfies the (primal) optimality conditions but it lies outside the feasible region of the primal
- Making it feasible is possible only at the price of decreasing the primal objective
- It is not the primal maximization problem that we are solving now but rather the dual minimization problem
- We do not need to rewrite the problem into the dual to apply the dual simplex method, it can run directly on the (primal) simplex tableau

The Dual Simplex Method: Example

- The new basis is dual feasible (primal optimal) but still not dual optimal, as $x_4 = -1 < 0$
- x_4 leaves the basis and $x_2 = \text{argmin}\{-\frac{z_j}{y_{4j}} : y_{4j} < 0\}$ enters

	x_1	x_2	x_3	x_4	x_5	RHS
z	0	0	$\frac{9}{5}$	$\frac{8}{5}$	$\frac{1}{5}$	$-\frac{28}{5}$
x_2	0	1	$-\frac{1}{5}$	$-\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
x_1	1	0	$\frac{7}{5}$	$-\frac{1}{5}$	$-\frac{2}{5}$	$\frac{11}{5}$

- The resultant basis is both dual optimal and dual feasible
- The optimum for the maximization problem is $z = -\frac{28}{5}$, attained at the point $\boldsymbol{x} = [\frac{11}{5} \quad \frac{2}{5} \quad 0]^T$

The Dual Simplex Method: Example

- Observe that the objective function value for the maximization problem has decreased in each iteration

$$\max -2x_1 - 3x_2 - 4x_3 : 0 \to -4 \to -\frac{28}{5}$$

- Of course, this is because we have in fact solved the dual minimization problem $\min\{\boldsymbol{w}^T \boldsymbol{b} : \boldsymbol{w}^T \boldsymbol{A} \geq \boldsymbol{c}^T\}$
- Choosing $\boldsymbol{w}^T = \boldsymbol{c_B} \boldsymbol{B}^{-1}$ the dual objective function $\boldsymbol{w}^T \boldsymbol{b} = \boldsymbol{c_B} \boldsymbol{B}^{-1} \boldsymbol{b}$ can be read from the simplex tableau in each step (row zero, RHS column)

$$\min \boldsymbol{w}^T \boldsymbol{b} : 0 \to -4 \to -\frac{28}{5}$$

- Originally we had a minimization problem (invert!), whose optimum is thus $z = \frac{28}{5}$

The Dual Simplex Method: Example

- Solve the below linear program:

$$
\begin{array}{lrcrcrcrcr}
\min & 2x_1 & + & 3x_2 & + & 5x_3 & + & 6x_4 & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & \geq & 2 \\
 & -2x_1 & + & x_2 & - & x_3 & + & 3x_4 & \leq & -3 \\
 & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0
\end{array}
$$

- Standard form, as a maximization (note: invert!)

$$
\begin{array}{lrcrcrcrcrcrcr}
\max & -2x_1 & - & 3x_2 & - & 5x_3 & - & 6x_4 & & & & & & \\
\text{s.t.} & x_1 & + & 2x_2 & + & 3x_3 & + & x_4 & - & x_5 & & & = & 2 \\
 & -2x_1 & + & x_2 & - & x_3 & + & 3x_4 & & & + & x_6 & = & -3 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0
\end{array}
$$

- Multiplying the first constraint by (-1) we obtain a primal optimal initial basis

The Dual Simplex Method: Example

- In general, slack variables constitute a primal feasible basis if $\boldsymbol{b} \geq \mathbf{0}$, and a dual feasible basis if $\boldsymbol{c}^T \leq \mathbf{0}$
- We can use the dual simplex now

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	2	3	5	6	0	0	0
x_5	0	-1	-2	-3	-1	1	0	-2
x_6	0	$\boxed{-2}$	1	-1	3	0	1	-3

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	4	4	9	0	1	-3
x_5	0	0	$-\frac{5}{2}$	$\boxed{-\frac{5}{2}}$	$-\frac{5}{2}$	1	$-\frac{1}{2}$	$-\frac{1}{2}$
x_1	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{3}{2}$	0	$-\frac{1}{2}$	$\frac{3}{2}$

The Dual Simplex Method: Example

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	0	0	5	$\frac{8}{5}$	$\frac{1}{5}$	$-\frac{19}{5}$
x_2	0	0	1	1	1	$-\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
x_1	0	1	0	1	-1	$-\frac{1}{5}$	$-\frac{2}{5}$	$\frac{8}{5}$

- The minimum is $\frac{19}{5}$, attained by the minimization problem at the point $\boldsymbol{x} = [\frac{8}{5} \quad \frac{1}{5} \quad 0 \quad 0]^T$
- The dual of the original minimization problem:

$$
\begin{array}{llcrcl}
\max & 2w_1 & - & 3w_2 & & \\
\text{s.t.} & w_1 & - & 2w_2 & \leq & 2 \\
 & 2w_1 & + & w_2 & \leq & 3 \\
 & 3w_1 & - & w_2 & \leq & 5 \\
 & w_1 & + & 3w_2 & \leq & 6 \\
 & w_1 & & & \geq & 0 \\
 & & & w_2, & \leq & 0
\end{array}
$$

The Dual Simplex Method: Example

- w_2 is odd since $w_2 \leq 0$ (the simplex requires nonnegativity)
- Let $w_2' = -w_2 \geq 0$

$$\begin{array}{lrcrcl} \max & 2w_1 & + & 3w_2' & & \\ \text{s.t.} & w_1 & + & 2w_2' & \leq & 2 \\ & 2w_1 & - & w_2' & \leq & 3 \\ & 3w_1 & + & w_2' & \leq & 5 \\ & w_1 & - & 3w_2' & \leq & 6 \\ & w_1, & & w_2' & \geq & 0 \end{array}$$

- The slack variables supply a primal feasible initial basis in the dual, since the RHS is nonnegatove and all constraints are of the type "$\leq$"
- Solving by the primal simplex: the optimum is $\frac{19}{5}$ and

$$w_1 = \frac{8}{5}, \qquad w_2 = -w_2' = -\frac{1}{5}$$

Optimal Employee Work Schedule

- At a railway station, the distribution of work is such that the number of staff needed is
  - 3 persons between 0 and 4 o’clock,
  - 8 persons between 4 and 8 o’clock,
  - 10 persons between 8 and 12 o’clock,
  - 8 persons between 12 and 16 o’clock,
  - 14 persons between 16 and 20 o’clock,
  - 5 persons between 20 and 24 o’clock
- Shifts start every day at 0, 4, 8, 12, 16, and 20 o’clock and keep 8 hours
- **Task:** obtain an optimal schedule that requires the smallest staff (fewest persons during the day working in total)

Optimal Employee Work Schedule

- Indicate the number of workers starting in each shift by x_1, x_2, x_3, x_4, x_5, and x_6
- Then, the task is to minimize the objective function $x_1 + x_2 + x_3 + x_4 + x_5 + x_6$
- From 0 until 4 o'clock, the 20-o'clock and 0-o-clock shifts are in work, at least 3 persons

$$x_1 + x_6 \geq 3$$

- From 4 until 8 o'clock at least 8 persons are needed

$$x_1 + x_2 \geq 8$$

- Similarly for the rest of the shifts
- Of course, all variables are nonnegative

Optimal Employee Work Schedule

- The linear program

$$
\begin{array}{lccccccccccccc}
\min & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & + & x_6 & & \\
\text{s.t.} & x_1 & + & x_2 & & & & & & & & & \geq & 8 \\
 & & & x_2 & + & x_3 & & & & & & & \geq & 10 \\
 & & & & & x_3 & + & x_4 & & & & & \geq & 8 \\
 & & & & & & & x_4 & + & x_5 & & & \geq & 14 \\
 & & & & & & & & & x_5 & + & x_6 & \geq & 5 \\
 & x_1 & & & & & & & & & + & x_6 & \geq & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0
\end{array}
$$

- Variables are continuous, even though we cannot schedule employees partially
- Should pose this as an integer linear program, by requiring variables to be integer-valued: NP-hard problem
- For now, we merely hope that what we obtain eventually will be integer-valued

Optimal Employee Work Schedule

- Introduce slack variables s_1, s_2, ... to convert the constraints $x_i + x_j \geq b$ to the form $x_i + x_j - s = b, z \geq 0$
- Writing as a maximization problem (note to ourselves: invert result at the end) and ignoring the column of z for now

	x_1	x_2	x_3	x_4	x_5	x_6	s_1	s_2	s_3	s_4	s_5	s_6	RHS
z	1	1	1	1	1	1	0	0	0	0	0	0	0
s_1	1	1	0	0	0	0	-1	0	0	0	0	0	8
s_2	0	1	1	0	0	0	0	-1	0	0	0	0	10
s_3	0	0	1	1	0	0	0	0	-1	0	0	0	8
s_4	0	0	0	1	1	0	0	0	0	-1	0	0	14
s_5	0	0	0	0	1	1	0	0	0	0	-1	0	5
s_6	1	0	0	0	0	1	0	0	0	0	0	-1	3

Optimal Employee Work Schedule

- Slack variables form a trivial initial basis (it is worth inverting all rows)
- Primal optimal basis but not primal feasible: use the dual simplex!
- s_4 leaves the basis and x_4 enters

	x_1	x_2	x_3	x_4	x_5	x_6	s_1	s_2	s_3	s_4	s_5	s_6	RHS
z	1	1	1	1	1	1	0	0	0	0	0	0	0
s_1	–1	–1	0	0	0	0	1	0	0	0	0	0	–8
s_2	0	–1	–1	0	0	0	0	1	0	0	0	0	–10
s_3	0	0	–1	–1	0	0	0	0	1	0	0	0	–8
s_4	0	0	0	$\boxed{-1}$	–1	0	0	0	0	1	0	0	–14
s_5	0	0	0	0	–1	–1	0	0	0	0	1	0	–5
s_6	–1	0	0	0	0	–1	0	0	0	0	0	1	–3

Optimal Employee Work Schedule

	x_1	x_2	x_3	x_4	x_5	x_6	s_1	s_2	s_3	s_4	s_5	s_6	RHS
z	1	1	1	0	0	1	0	0	0	1	0	0	–14
s_1	–1	–1	0	0	0	0	1	0	0	0	0	0	–8
s_2	0	$\boxed{-1}$	–1	0	0	0	0	1	0	0	0	0	–10
s_3	0	0	–1	0	1	0	0	0	1	–1	0	0	6
x_4	0	0	0	1	1	0	0	0	0	–1	0	0	14
s_5	0	0	0	0	–1	–1	0	0	0	0	1	0	–5
s_6	–1	0	0	0	0	–1	0	0	0	0	0	1	–3

- s_2 leaves the basis and x_2 enters

Optimal Employee Work Schedule

	x_1	x_2	x_3	x_4	x_5	x_6	s_1	s_2	s_3	s_4	s_5	s_6	RHS
z	1	0	0	0	0	1	0	1	0	1	0	0	–24
s_1	–1	0	1	0	0	0	1	–1	0	0	0	0	2
x_2	0	1	1	0	0	0	0	–1	0	0	0	0	10
s_3	0	0	–1	0	1	0	0	0	1	–1	0	0	6
x_4	0	0	0	1	1	0	0	0	0	–1	0	0	14
s_5	0	0	0	0	[–1]	–1	0	0	0	0	1	0	–5
s_6	–1	0	0	0	0	–1	0	0	0	0	0	1	–3

- s_5 leaves, x_5 enters
- Dual degenerate pivot

Optimal Employee Work Schedule

	x_1	x_2	x_3	x_4	x_5	x_6	s_1	s_2	s_3	s_4	s_5	s_6	RHS
z	1	0	0	0	0	1	0	1	0	1	0	0	−24
s_1	−1	0	1	0	0	0	1	−1	0	0	0	0	2
x_2	0	1	1	0	0	0	0	−1	0	0	0	0	10
s_3	0	0	−1	0	0	−1	0	0	1	−1	1	0	1
x_4	0	0	0	1	0	−1	0	0	0	−1	1	0	9
x_5	0	0	0	0	1	1	0	0	0	0	−1	0	5
s_6	**−1**	0	0	0	0	−1	0	0	0	0	0	1	−3

- s_6 leaves, x_1 enters

Optimal Employee Work Schedule

	x_1	x_2	x_3	x_4	x_5	x_6	s_1	s_2	s_3	s_4	s_5	s_6	RHS
z	0	0	0	0	0	0	0	1	0	1	0	1	−27
s_1	0	0	1	0	0	1	1	−1	0	0	0	−1	5
x_2	0	1	1	0	0	0	0	−1	0	0	0	0	10
s_3	0	0	−1	0	0	−1	0	0	1	−1	1	0	1
x_4	0	0	0	1	0	−1	0	0	0	−1	1	0	9
x_5	0	0	0	0	1	1	0	0	0	0	−1	0	5
x_1	1	0	0	0	0	1	0	0	0	0	0	−1	3

- Optimal tableau: primal optimal and now also primal feasible
- 27 persons employed in total: 3 in the 0 o'clock shift, 10 in the 4 o'clock shift, 9 in the 12 o'clock shift, and 5 in the 16 o'clock shift, and no one works in the rest of the shifts
- Observe that there is surplus staff from 4 until 8 ($s_1 = 5$) and from 12 until 16 ($s_3 = 1$)!

Optimal Production Planning

- The four-month prognosis for an item in a store is as follows:

	1st month	2nd month	3rd month	4th month
Business Plan [t]	5	6	8	6
Purchase cost [mUSD/t]	4	3	2	5
Storage capacity [t]	10	10	10	10
Storage costs [mUSD/t]	1.5	1.5	1.5	1.5

- At the beginning and end of the period the stock in the storage is zero
- The stock changes uniformly during a month and the storage cost per month is based on the average quantity in stock
- **Task:** fulfill the business plan in each month, taking into account the storage capacities, with the lowest purchase and storage costs

Optimal Production Planning

- Denote the quantity of the items purchased in each month by x_1, x_2, x_3, and x_4 [tonnes]
- Denote the stock at the end of each month by r_1, r_2, and r_3 [tonnes] (no stock at the end of the period)
- From the quantity purchased in the first month, 5 tonnes must be sold according to the business plan and the rest goes into stock

$$x_1 = 5 + r_1$$

- In the rest of the months, the stock at the beginning plus the purchased quantity covers the monthly business plan and the stock at the end of the month

$$r_1 + x_2 = 6 + r_2$$

$$r_2 + x_3 = 8 + r_3$$

$$r_3 + x_4 = 6$$

Optimal Production Planning

- As the stock changes uniformly during the month, the average stock in each month is

$$\frac{r_1}{2}, \quad \frac{r_1 + r_2}{2}, \quad \frac{r_2 + r_3}{2}, \quad \frac{r_3}{2}$$

- The storage cost for the entire period [million USD]:

$$1.5\left(\frac{r_1}{2} + \frac{r_1 + r_2}{2} + \frac{r_2 + r_3}{2} + \frac{r_3}{2}\right) = 1.5r_1 + 1.5r_2 + 1.5r_3$$

- The purchase cost: $4x_1 + 3x_2 + 2x_3 + 5x_4$ [million USD]
- Finally, the stock cannot exceed the storage capacity:

$$r_1, r_2, r_3 \leq 10$$

- Evidently, all variables are nonnegative

Optimal Production Planning

- The linear program:

$$\begin{array}{lccccccccl}
\min & 4x_1 & +3x_2 & +2x_3 & +5x_4 & +1.5r_1 & +1.5r_2 & +1.5r_3 & \\
\text{s.t.} & x_1 & & & & -r_1 & & & = 5 \\
& & x_2 & & & +r_1 & -r_2 & & = 6 \\
& & & x_3 & & & +r_2 & -r_3 & = 8 \\
& & & & x_4 & & & +r_3 & = 6 \\
& & & & & r_1 & & & \leq 10 \\
& & & & & & r_2 & & \leq 10 \\
& & & & & & & r_3 & \leq 10 \\
& x_1, & x_2, & x_3, & x_4, & r_1, & r_2, & r_3 & \geq 0
\end{array}$$

- The objective function in maximization form:

$$\max \quad -4x_1 \quad -3x_2 \quad -2x_3 \quad -5x_4 \quad -1.5r_1 \quad -1.5r_2 \quad -1.5r_3$$

- Must be inverted when written into the simplex tableau!

Optimal Production Planning

- We still need to find an initial basis
  - the slacks for the storage constraints (s_1, s_2, s_3) are OK
  - x_1, x_2, x_3, and x_4 would also work, but we first need to zero out the corresponding objective function coefficients to obtain a valid simplex tableau

	z	x_1	x_2	x_3	x_4	r_1	r_2	r_3	s_1	s_2	s_3	RHS
z	1	4	3	2	5	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	0	0	0	0
x_1	0	1	0	0	0	-1	0	0	0	0	0	5
x_2	0	0	1	0	0	1	-1	0	0	0	0	6
x_3	0	0	0	1	0	0	1	-1	0	0	0	8
x_4	0	0	0	0	1	0	0	1	0	0	0	6
s_1	0	0	0	0	0	1	0	0	1	0	0	10
s_2	0	0	0	0	0	0	1	0	0	1	0	10
s_3	0	0	0	0	0	0	0	1	0	0	1	10

Optimal Production Planning

- Subtract four times the row of x_1 from row 0, this way eliminating the reduced cost for x_1:

	z	x_1	x_2	x_3	x_4	r_1	r_2	r_3	s_1	s_2	s_3	RHS
z	1	0	3	2	5	$\frac{11}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	0	0	0	-20
x_1	0	1	0	0	0	-1	0	0	0	0	0	5
x_2	0	0	1	0	0	1	-1	0	0	0	0	6
x_3	0	0	0	1	0	0	1	-1	0	0	0	8
x_4	0	0	0	0	1	0	0	1	0	0	0	6
s_1	0	0	0	0	0	1	0	0	1	0	0	10
s_2	0	0	0	0	0	0	1	0	0	1	0	10
s_3	0	0	0	0	0	0	0	1	0	0	1	10

- Note how the objective function value has changed!

Optimal Production Planning

- Similarly, cancel the reduced cost for x_2, x_3, and x_4 by elementary row operations

	z	x_1	x_2	x_3	x_4	r_1	r_2	r_3	s_1	s_2	s_3	RHS
z	1	0	0	0	0	$\frac{5}{2}$	$\frac{5}{2}$	$-\frac{3}{2}$	0	0	0	-84
x_1	0	1	0	0	0	-1	0	0	0	0	0	5
x_2	0	0	1	0	0	1	-1	0	0	0	0	6
x_3	0	0	0	1	0	0	1	-1	0	0	0	8
x_4	0	0	0	0	1	0	0	$\boxed{1}$	0	0	0	6
s_1	0	0	0	0	0	1	0	0	1	0	0	10
s_2	0	0	0	0	0	0	1	0	0	1	0	10
s_3	0	0	0	0	0	0	0	1	0	0	1	10

- Primal feasible tableau, solve with the primal simplex
- Note the nonzero objective function in the initial tableau!

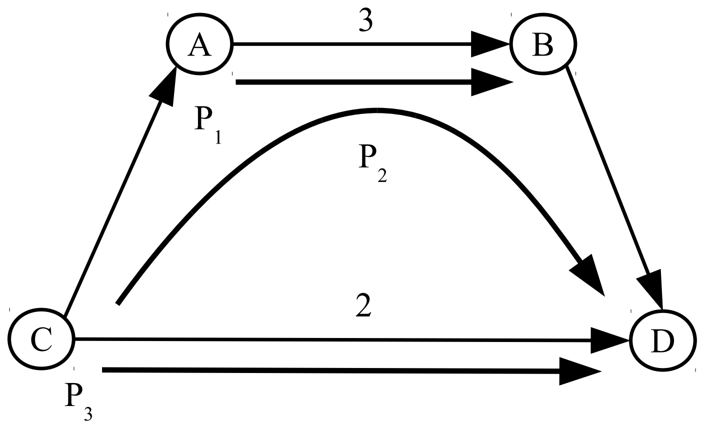
Optimal Production Planning

	z	x_1	x_2	x_3	x_4	r_1	r_2	r_3	s_1	s_2	s_3	RHS
z	1	0	0	0	$\frac{3}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	0	0	0	0	-75
x_1	0	1	0	0	0	-1	0	0	0	0	0	5
x_2	0	0	1	0	0	1	-1	0	0	0	0	6
x_3	0	0	0	1	1	0	1	0	0	0	0	14
r_3	0	0	0	0	1	0	0	1	0	0	0	6
s_1	0	0	0	0	0	1	0	0	1	0	0	10
s_2	0	0	0	0	0	0	1	0	0	1	0	10
s_3	0	0	0	0	-1	0	0	0	0	0	1	4

- The quantity of items to be purchased in each month is 5, 6, and 14 tonnes, no purchase in the last month
- Stock is created only in the 3rd month, 6 tonnes
- The total cost is 75 million USD

Network Routing

- A subscriber wishes to transfer 2-2 units of traffic between points A-B and C-D in a telecommunications network
- The network service provider establishes a path P_1 between A-B and paths P_2 and P_3 between C-D
- The A-B link capacity is 3 units, and 2 units for C-D
- Pricing is progressive: the first 1 unit of traffic through a link costs 1 unit, every additional unit costs 3 units



- **Task:** find the minimal cost assignment of traffic demands to paths

Network Routing

- Denote the quantity of traffic routed to paths P_1, P_2, and P_3 by f_1, f_2, and f_3
- The demand is 2 units of flow between A-B and 2 units between C-D

$$f_1 \geq 2, \qquad f_2 + f_3 \geq 2$$

- Denote the total load at each link by l_1 and l_2, these must satisfy the capacity constraints

$$l_1 = f_1 + f_2 \leq 3, \qquad l_2 = f_3 \leq 2$$

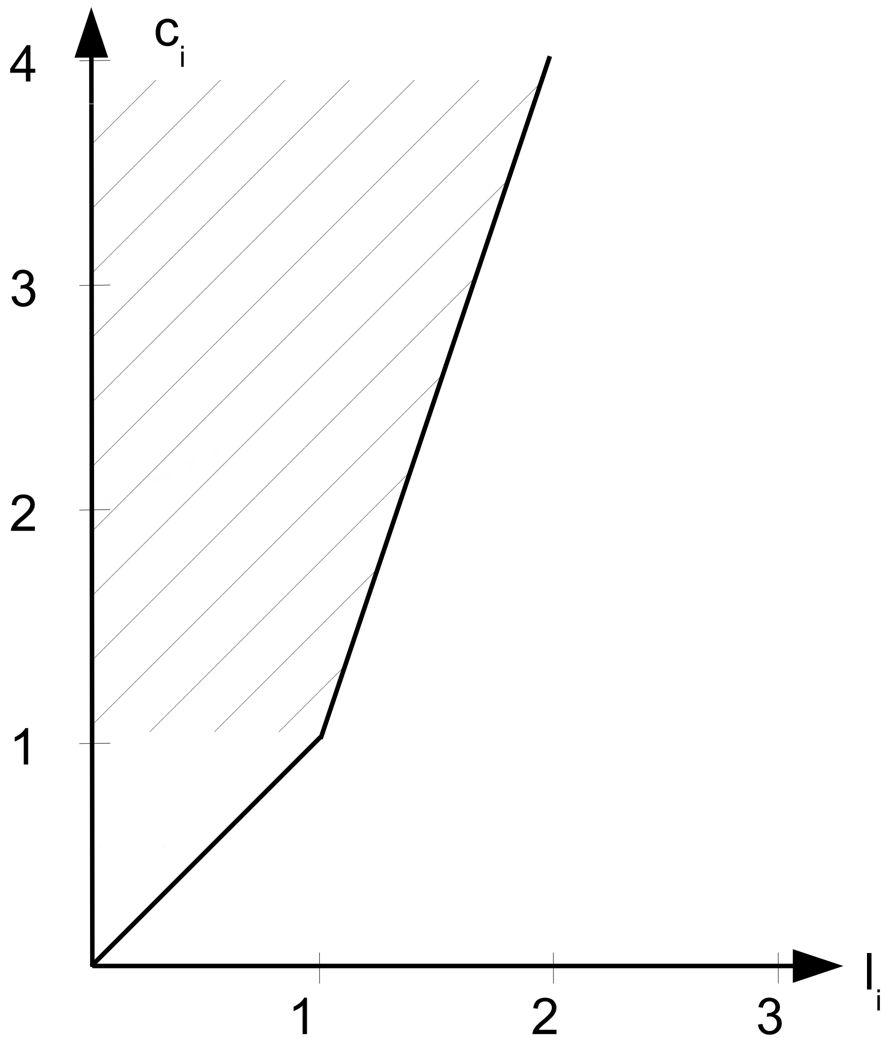
- Let the price of traffic routed to each link be c_1 and c_2

$$c_i = \begin{cases} l_i & \text{if } l_i \leq 1 \\ 1 + 3(l_i - 1) & \text{if } l_i > 1 \end{cases} \qquad i \in \{1, 2\}$$

- Task is to minimize $c_1 + c_2$: nonlinear objective function!

Network Routing

- Trick: linearize the objective function



- Piecewise linear objective function
- Approximate piecewise

$$\min c_i$$

$$c_i \geq l_i$$

$$c_i \geq 3l_i - 2$$

- The smallest possible cost by minimization
- The piecewise objective is convex: we can use linear programming

Network Routing

- The linear program:

$$
\begin{aligned}
\min \quad & c_1 + c_2 & \\
\text{s.t.} \quad & f_1 + f_2 & \leq 3 \\
& f_3 & \leq 2 \\
& f_1 + f_2 - c_1 & \leq 0 \\
& 3f_1 + 3f_2 - c_1 & \leq 2 \\
& f_3 - c_2 & \leq 0 \\
& 3f_3 - c_2 & \leq 2 \\
& f_1 & \geq 2 \\
& f_2 + f_3 & \geq 2 \\
& f_1, f_2, f_3, c_1, c_2, & \geq 0
\end{aligned}
$$

Network Routing

- Standard form: slack variables constitute an initial basis
- Converting to maximization and inverting the last two constraints we get a dual feasible initial basis
- Use the dual simplex!

	z	f_1	f_2	f_3	c_1	c_2	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	RHS
z	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0
s_1	0	1	1	0	0	0	1	0	0	0	0	0	0	0	3
s_2	0	0	0	1	0	0	0	1	0	0	0	0	0	0	2
s_3	0	1	1	0	-1	0	0	0	1	0	0	0	0	0	0
s_4	0	3	3	0	-1	0	0	0	0	1	0	0	0	0	2
s_5	0	0	0	1	0	-1	0	0	0	0	1	0	0	0	0
s_6	0	0	0	3	0	-1	0	0	0	0	0	1	0	0	2
s_7	0	-1	0	0	0	0	0	0	0	0	0	0	1	0	-2
s_8	0	0	-1	-1	0	0	0	0	0	0	0	0	0	1	-2

Network Routing

- The optimal tableau:

	z	f_1	f_2	f_3	c_1	c_2	s_1	s_2	s_3	s_4	s_5	s_6	s_7	s_8	RHS
z	1	0	0	0	0	0	0	0	0	1	0	1	3	3	-8
s_1	0	0	0	0	0	0	1	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	1	1	0
s_2	0	0	0	0	0	0	0	1	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	0	1
s_3	0	0	0	0	0	0	0	0	1	-1	1	-1	-2	-2	4
c_1	0	0	0	0	1	0	0	0	0	-1	$\frac{3}{2}$	$-\frac{3}{2}$	-3	-3	7
c_2	0	0	0	0	0	1	0	0	0	0	$-\frac{3}{2}$	$\frac{1}{2}$	0	0	1
f_2	0	0	1	0	0	0	0	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	0	-1	1
f_1	0	1	0	0	0	0	0	0	0	0	0	0	-1	0	2
f_3	0	0	0	1	0	0	0	0	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	1

- Route 2 unit to path P_1 and route 1 unit to each of the paths P_2 and P_3
- The total cost is 8 units