# Duality in Linear Programming: A Summary

*WARNING: this is just a summary of the material covered in the full slide-deck **Duality in Linear Programming** that will orient you as per the topics covered there; you are required to learn the full version, not just this summary!*

- The dual of a linear program: motivation
- Formal definition, the Karush-Kuhn-Tucker (KKT) conditions (optional)
- Primal–dual relationships, the Weak and the Strong Duality Theorems
- The Farkas Lemma (optional)

# Linear Programming Duality: Motivation

- Consider the below linear program

$$
\begin{array}{rrrrrrrrrr}
z & = & \max & x_1 & + & 2x_2 & - & x_3 & & \\
& & \text{s.t.} & 3x_1 & + & 2x_2 & + & x_3 & \leq & 12 \\
& & & -x_1 & & & - & x_3 & \leq & -3 \\
& & & x_1, & & x_2, & & x_3 & \geq & 0
\end{array}
$$

- We give upper bounds for the objective function
- Since variables are nonnegative, the first constraint is immediately an upper bound

$$z = x_1 + 2x_2 - x_3 \leq 3x_1 + 2x_2 + x_3 \leq 12$$

- Since component-wise $x_1 \leq 3x_1$, $2x_2 \leq 2x_2$, and $-x_3 \leq x_3$
- Is there any tighter upper bound?

# Linear Programming Duality: Motivation

- Summing the two constraints

$$
\begin{array}{cccccccc}
z = & x_1 & + & 2x_2 & - & x_3 & & \\
\hline
 & 3x_1 & + & 2x_2 & + & x_3 & \leq & 12 \\
\oplus & -x_1 & & & - & x_3 & \leq & -3 \\
\hline
 & 2x_1 & + & 2x_2 & + & 0x_3 & \leq & 9
\end{array}
$$

- Yields the tighter bound $z = x_1 + 2x_2 - x_3 \leq 2x_1 + 2x_2 \leq 9$
- Even tighter bound is obtained if we add two times the second constraint to the first one:

$$z = x_1 + 2x_2 - x_3 \leq (3-2*1)x_1 + (2-2*0)x_2 + (1-2*1)x_3 \leq 6$$

- This is the tightest possible bound, since the optimal objective function value is $z = 6$

# Linear Programming Duality: Motivation

- In fact, for any $w_1 \geq 0$ and $w_2 \geq 0$ for which the expression

$$w_1\left(3x_1 + 2x_2 + x_3\right) + w_2\left(-x_1 - x_3\right) \leq 12w_1 - 3w_2$$

  component-wise upper bounds the objective function $z = x_1 + 2x_2 - x_3$, that is, for which

$$3w_1 - w_2 \geq 1, \quad 2w_1 \geq 2, \quad \text{and} \quad w_1 - w_2 \geq -1$$

  holds, we get a new upper bound:

$$z = x_1 + 2x_2 - x_3 \leq w_1\left(3x_1 + 2x_2 + x_3\right) + w_2\left(-x_1 - x_3\right)$$

- $w_1 \geq 0$ and $w_2 \geq 0$ needed, otherwise the sign would change
- The tightest bound is the one for which $12w_1 + (-3)w_2$ is minimal

# Linear Programming Duality: Motivation

- Yields another linear program: the **dual linear program**:

$$
\begin{array}{lrcrcr}
\min & 12w_1 & - & 3w_2 & & \\
\text{s.t.} & 3w_1 & - & w_2 & \geq & 1 \\
 & 2w_1 & & & \geq & 2 \\
 & w_1 & - & w_2 & \geq & -1 \\
 & w_1, & & w_2 & \geq & 0
\end{array}
$$

- To distinguish, the original linear program will be called the **primal**
- For the primal $\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ we get the dual $\min\{\boldsymbol{w}^T\boldsymbol{b} : \boldsymbol{w}^T\boldsymbol{A} \geq \boldsymbol{c}^T, \boldsymbol{w}^T \geq \boldsymbol{0}\}$
- Interestingly, the dual optimal solution is also $6$
- In fact this is guaranteed to hold and, what is more, there are very deep relationships between the primal and the dual

# The Dual Linear Program

- **Theorem:** given a linear program as a maximization problem in the **standard form**

$$\begin{aligned} \max\ & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t. } & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

  the dual is the standard form minimization problem

$$\begin{aligned} \min\ & \boldsymbol{w}^T \boldsymbol{b} \\ \text{s.t. } & \boldsymbol{w}^T \boldsymbol{A} - \boldsymbol{v}^T = \boldsymbol{c}^T \\ & \boldsymbol{v}^T \geq \boldsymbol{0},\ \boldsymbol{w}^T \text{ arbitrary} \end{aligned}$$

- One dual variable for each constraint of the primal and one dual constraint for each variable of the primal

# The Dual Linear Program

- The variables $\boldsymbol{v}^T$ and $\boldsymbol{w}^T$ are called **dual variables** (or Lagrangean multipliers)
- The dual variables $\boldsymbol{w}^T = [w_1 \quad w_2 \ \dots \quad w_m]$ correspond to the primal constraints $\boldsymbol{Ax} = \boldsymbol{b}$: for every constraint $\boldsymbol{a^i x} = b_i$ there is a dual variable $w_i$, precisely $m$
- The dual variables $\boldsymbol{v}^T = [v_1 \quad v_2 \ \dots \quad v_n]$ correspond to the nonnegativity constraints for the primal variables $\boldsymbol{x}$: for every constraint $x_j \geq 0$ there is a dual variable $v_j$, exactly $n$
- In fact, $\boldsymbol{v}^T$ act as slack-variables that we can as well omit

$$\begin{aligned} \min\ & \boldsymbol{w}^T \boldsymbol{b} \\ \text{s.t. } & \boldsymbol{w}^T \boldsymbol{A} \geq \boldsymbol{c}^T \\ & \boldsymbol{w}^T \text{ arbitrary} \end{aligned}$$

# The Primal and the Dual: Canonical Form

- In general, the primal and dual in canonical form:

$$
\begin{aligned}
P:\ \max\ & \boldsymbol{c}^T\boldsymbol{x} \\
\text{s.t. } & \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b} \\
& \boldsymbol{x} \ge \boldsymbol{0}
\end{aligned}
\qquad\qquad
\begin{aligned}
D:\ \min\ & \boldsymbol{w}^T\boldsymbol{b} \\
\text{s.t.} & \boldsymbol{w}^T\boldsymbol{A} \ge \boldsymbol{c}^T \\
& \boldsymbol{w}^T \ge \boldsymbol{0}
\end{aligned}
$$

# The Dual Linear Program

| | Maximization problem | | Minimization problem | |
|---|---|---|---|---|
| Constraint | $\geq$ | $\longleftrightarrow$ | $\leq 0$ | Variable |
| | $\leq$ | $\longleftrightarrow$ | $\geq 0$ | |
| | $=$ | $\longleftrightarrow$ | arbitrary | |
| Variable | $\geq 0$ | $\longleftrightarrow$ | $\geq$ | Constraint |
| | $\leq 0$ | $\longleftrightarrow$ | $\leq$ | |
| | arbitrary | $\longleftrightarrow$ | $=$ | |

# The Dual Linear Program

| | Primal | Dual |
|---|---|---|
| Standard form | $\max \boldsymbol{c}^T\boldsymbol{x}$<br>s.t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$<br>$\boldsymbol{x} \geq \boldsymbol{0}$ | $\min \boldsymbol{w}^T\boldsymbol{b}$<br>s.t. $\boldsymbol{w}^T\boldsymbol{A} \geq \boldsymbol{c}^T$<br>$\boldsymbol{w}^T$ arbitrary |
| Canonical form | $\max \boldsymbol{c}^T\boldsymbol{x}$<br>s.t. $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$<br>$\boldsymbol{x} \geq \boldsymbol{0}$ | $\min \boldsymbol{w}^T\boldsymbol{b}$<br>s.t. $\boldsymbol{w}^T\boldsymbol{A} \geq \boldsymbol{c}^T$<br>$\boldsymbol{w}^T \geq \boldsymbol{0}$ |

# Primal–Dual Relationships

- **Theorem:** the dual of the dual linear program is the primal
- **The Weak Duality Theorem:** the objective function value for *any* feasible solution for the primal maximization problem is less than, or equal to the objective function value for *any* feasible solution for the dual minimization problem
- Note the importance of the *any* quantification: *any* primal-feasible $\boldsymbol{x}$ gives a lower bound $\boldsymbol{c}^T\boldsymbol{x}$ for the dual, and of course *any* dual-feasible $\boldsymbol{w}^T$ gives an upper bound $\boldsymbol{w}^T\boldsymbol{b}$ for the primal
- **Corollaries:**
  - if $\boldsymbol{x}$ is primal-feasible, $\boldsymbol{w}^T$ is dual-feasible, and $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{w}^T\boldsymbol{b}$, then $\boldsymbol{x}$ is optimal in the primal and $\boldsymbol{w}^T$ is optimal in the dual
  - if the primal is unbounded then the dual is infeasible and *vice versa*

# Primal–Dual Relationships

- **The Strong Duality Theorem:** for the primal–dual pair of linear programs exactly one of the below claims holds true
  - the primal has an optimal solution $\bar{\boldsymbol{x}}$ and the dual also has an optimal solution $\bar{\boldsymbol{w}}^T$, and $\boldsymbol{c}^T\bar{\boldsymbol{x}} = \bar{\boldsymbol{w}}^T\boldsymbol{b}$
  - one of the problems is unbounded and therefore the other is infeasible
  - neither problem is feasible

| | | |
|---|---|---|
| P optimal | $\Longleftrightarrow$ | D optimal |
| P unbounded | $\Longrightarrow$ | D infeasible |
| D unbounded | $\Longrightarrow$ | P infeasible |
| P infeasible | $\Longrightarrow$ | D unbounded or infeasible |
| D infeasible | $\Longrightarrow$ | P unbounded or infeasible |

# Duality: Example

- Solve the below linear program

$$\begin{array}{lrcrcrcr}
\max & -5x_1 & - & 2x_2 & - & x_3 & & \\
\text{s.t.} & -x_1 & - & 2x_2 & & & \leq & 1 \\
& -2x_1 & - & 2x_2 & & & \leq & 3 \\
& -5x_1 & + & x_2 & - & x_3 & \leq & -5 \\
& 5x_1 & + & 3x_2 & - & x_3 & \leq & -2 \\
& x_1, & & x_2, & & x_3 & \geq & 0
\end{array}$$

- In standard form:

$$\begin{array}{lccccccccc}
\max & -5x_1 & -2x_2 & -x_3 & & & & & & \\
\text{s.t.} & -x_1 & -2x_2 & & +x_4 & & & & = & 1 \\
& -2x_1 & -2x_2 & & & +x_5 & & & = & 3 \\
& -5x_1 & +x_2 & -x_3 & & & +x_6 & & = & -5 \\
& 5x_1 & +3x_2 & -x_3 & & & & +x_7 & = & -2 \\
& x_1, & x_2, & x_3, & x_4, & x_5, & x_6, & x_7 & \geq & 0
\end{array}$$

# Duality: Example

- Find an initial feasible basis
- The trivial choice would be to choose the columns of the slack variables into the initial basis, in particular if $B = \{x_4, x_5, x_6, x_7\}$ then $\boldsymbol{B} = \boldsymbol{B}^{-1} = \boldsymbol{I}_4$
- Unfortunately, this trivial basis is not (primal) feasible, since $\bar{\boldsymbol{b}} = \boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{b} \ngeq \mathbf{0}$
- Let us write the dual, in the hope that it will be easier to find an initial basis for that

$$
\begin{array}{lrcrcrcrcr}
\min & w_1 & + & 3w_2 & - & 5w_3 & - & 2w_4 & & \\
\text{s.t.} & -w_1 & - & 2w_2 & - & 5w_3 & + & 5w_4 & \geq & -5 \\
 & -2w_1 & - & 2w_2 & + & w_3 & + & 3w_4 & \geq & -2 \\
 & & & & - & w_3 & - & w_4 & \geq & -1 \\
 & w_1, & & w_2, & & w_3, & & w_4 & \geq & 0
\end{array}
$$

# Duality: Example

- Converting to standard form and rewriting the objective as a maximization problem (note to ourselves: we'll need to invert the resultant objective function due to the $\min \Rightarrow \max$ conversion!)

$$
\begin{array}{lcccccccc}
\max & -w_1 & -3w_2 & +5w_3 & +2w_4 & & & & \\
\text{s.t.} & -w_1 & -2w_2 & -5w_3 & +5w_4 & -w_5 & & & = -5 \\
 & -2w_1 & -2w_2 & +w_3 & +3w_4 & & -w_6 & & = -2 \\
 & & & -w_3 & -w_4 & & & -w_7 & = -1 \\
 & w_1, & w_2, & w_3 & w_4, & w_5, & w_6, & w_7 & \geq \; 0
\end{array}
$$

- The slack variables form an initial feasible basis, as

$$\boldsymbol{B} = \boldsymbol{B}^{-1} = -\boldsymbol{I}_3 \text{ and } \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \geq \boldsymbol{0}$$

- Use the (primal) simplex from here (invert all constraints!)

# Duality: Example

- The initial simplex tableau:

| | $z$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $w_7$ | RHS |
|---|---|---|---|---|---|---|---|---|---|
| $z$ | 1 | 1 | 3 | $-5$ | $-2$ | 0 | 0 | 0 | 0 |
| $w_5$ | 0 | 1 | 2 | $\boxed{5}$ | $-5$ | 1 | 0 | 0 | 5 |
| $w_6$ | 0 | 2 | 2 | $-1$ | $-3$ | 0 | 1 | 0 | 2 |
| $w_7$ | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 |

- Recall the pivot rules
  - optimality condition: $z_k = \min_{j \in N} z_j \geq 0$
  - $k$ enters the basis, if $k = \operatorname{argmin}_{j \in N} z_j$
  - $r$ leaves the basis, if $r = \underset{i \in \{1,\ldots,m\}}{\operatorname{argmin}} \left\{ \dfrac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$
- So $w_3$ enters and $w_5$ (or $w_7$) leaves the basis

# Duality: Example

- After the first pivot

| | $z$ | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $w_7$ | RHS |
|---|---|---|---|---|---|---|---|---|---|
| $z$ | 1 | 2 | 5 | 0 | $-7$ | 1 | 0 | 0 | 5 |
| $w_3$ | 0 | $\frac{1}{5}$ | $\frac{2}{5}$ | 1 | $-1$ | $\frac{1}{5}$ | 0 | 0 | 1 |
| $w_6$ | 0 | $\frac{11}{5}$ | $\frac{12}{5}$ | 0 | $-4$ | $\frac{1}{5}$ | 1 | 0 | 3 |
| $w_7$ | 0 | $-\frac{1}{5}$ | $-\frac{2}{5}$ | 0 | $\boxed{2}$ | $-\frac{1}{5}$ | 0 | 1 | 0 |

- $w_4$ enters and $w_7$ leaves the basis, and so on
- The optimal dual solution: $\boldsymbol{w}^T = [0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 3 \quad 0]$
- The optimal objective function value is $-5$, since we must invert the result due to the $\min \Rightarrow \max$ objective function conversion
- This is the optimum of the primal as well (Strong Theorem)