The Simplex Tableau: A Summary

*WARNING: this is just a summary of the material covered in the full slide-deck **The Simplex Tableau** that will orient you as per the topics covered there; you are required to learn the full version, not just this summary!*

- Recall: basic feasible solutions and the simplex pivot
- Termination: optimality and unbounded optimal solutions
- The steps of the simplex method
- Degeneration and cycling
- Complexity (worst-case and practical)
- The simplex tableau
- Solving linear programs using the simplex tableau: examples

Recall: The Simplex Method

- Let $\boldsymbol{A}$ be an $m \times n$ matrix with $\mathrm{rank}(\boldsymbol{A}) = \mathrm{rank}(\boldsymbol{A}, \boldsymbol{b}) = m$, $\boldsymbol{b}$ be a column m-vector, $\boldsymbol{x}$ be a column n-vector, and $\boldsymbol{c}^T$ be a row n-vector, and consider the linear program

$$\begin{aligned} z = \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- Let $\boldsymbol{B}$ be a basis and reorder the columns of $\boldsymbol{A}$ to obtain $\boldsymbol{A} = [\boldsymbol{B} \quad \boldsymbol{N}]$

- Furthermore, let $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_{\boldsymbol{B}} \\ \boldsymbol{x}_{\boldsymbol{N}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}$ the basic solution generated by $\boldsymbol{B}$ and suppose that this basic solution is feasible ($\boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$)

Recall: The Simplex Method

- The linear program in the nonbasic variable space:

$$\begin{aligned} \max \quad & z_0 + \textstyle\sum_{j \in N} z_j x_j \\ \text{s.t.} \quad & \boldsymbol{x_B} = \bar{\boldsymbol{b}} - \textstyle\sum_{j \in N} \boldsymbol{y}_j x_j \\ & \boldsymbol{x_B}, \boldsymbol{x_N} \geq \mathbf{0} \end{aligned}$$

  where
  - N denotes the set of nonbasic variables
  - $\bar{\boldsymbol{b}} = \boldsymbol{B}^{-1}\boldsymbol{b}$
  - $\boldsymbol{y}_j$ denotes the column of the matrix $\boldsymbol{B}^{-1}\boldsymbol{N}$ that belongs to the j-th nonbasic variable: $\boldsymbol{y}_j = \boldsymbol{B}^{-1}\boldsymbol{a}_j$
  - $z_0 = \boldsymbol{c_B}^T \mathrm{B}^{-1}\boldsymbol{b} = \boldsymbol{c_B}^T \bar{\boldsymbol{b}}$
  - z_j is the component of the row vector $\boldsymbol{c_N}^T - \boldsymbol{c_B}^T \boldsymbol{B}^{-1}\boldsymbol{N}$ that belongs to the j-th nonbasic variable

Recall: The Simplex Method

- **Pivot:** increase a nonbasic variable that improves the objective function until a basic variable drops to zero, and leave all other nonbasic variables unchanged
- Pivot rules
  - x_k can enter the basis if $z_k > 0$
  - x_r leaves the basis where $r = \operatorname*{argmin}_{i \in \{1,\ldots,m\}} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$
- The optimality condition of the (primal) simplex method: the basic feasible solution $\bar{\boldsymbol{x}}$ is optimal if

$$\forall j \in N : z_j \leq 0$$

Termination with Unboundedness

- Recall, if for some nonbasic variable x_k: $z_k > 0$, then increasing x_k increases the objective function
- We can keep on increasing x_k until some basic variable drops to zero:

$$x_k \leq \min_{i \in B} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$$

- If no such basic variable exists, then no basic variable blocks the growth of x_k
- **Theorem:** the optimal solution of the linear program $\max\{\boldsymbol{c}^T \boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ is **unbounded** if there is basic feasible solution $\bar{\boldsymbol{x}}$ and nonbasic variable x_k so that $z_k > 0$ and $\boldsymbol{y}_k \leq 0$

The Simplex Method: Initialization

- Let $\boldsymbol{A}$ be an $m \times n$ matrix with $\text{rank}(\boldsymbol{A}) = \text{rank}(\boldsymbol{A}, \boldsymbol{b}) = m$, $\boldsymbol{b}$ be a column m-vector, $\boldsymbol{x}$ be a column n-vector, and $\boldsymbol{c}^T$ be a row n-vector, and consider the linear program

$$\begin{aligned} z = \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- Suppose that all basic feasible solutions are nondegenerate
- The simplex method is an iterative algorithm to solve the above linear program, which uses nothing else than a subroutine to solve systems of linear equations and basic linear algebra operations
- **Initialization:** find an initial basic feasible solution and the corresponding basis $\boldsymbol{B}$ (see later on how to do this)

The Simplex Method: Main Step

1. Solve the system $\boldsymbol{B}\boldsymbol{x_B} = \boldsymbol{b}$

- The solution is unique: $\boldsymbol{x_B} = \boldsymbol{B}^{-1}\boldsymbol{b} = \bar{\boldsymbol{b}}$. Let $\boldsymbol{x_N} = \boldsymbol{0}$

2. Solve the system $\boldsymbol{w}^T\boldsymbol{B} = \boldsymbol{c_B}^T$

- The solution is unique: $\boldsymbol{w}^T = \boldsymbol{c_B}^T\boldsymbol{B}^{-1}$
- For each nonbasic variable j obtain the **reduced cost** $z_j = c_j - \boldsymbol{w}^T\boldsymbol{a_j}$ and choose the entering variable as

$$k = \operatorname*{argmax}_{j \in N} z_j \qquad \text{(Dantzig's pivot rule)}$$

3. If $z_k \leq 0$ then terminate: $\begin{bmatrix} \boldsymbol{x_B} \\ \boldsymbol{x_N} \end{bmatrix}$ is an optimal solution and the optimal objective function value is $\boldsymbol{c_B}^T\boldsymbol{x_B}$

- Otherwise proceed to the next step

The Simplex Method: Main Step

4. Solve the system $\boldsymbol{B}\boldsymbol{y}_k = \boldsymbol{a_k}$

- The solution is unique: $\boldsymbol{y}_k = \boldsymbol{B}^{-1}\boldsymbol{a_k}$
- If $\boldsymbol{y}_k \leq \boldsymbol{0}$ then terminate: the linear program is unbounded along the ray $\left\{ \begin{bmatrix} \bar{\boldsymbol{b}} \\ \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} -\boldsymbol{y}_k \\ \boldsymbol{e_k} \end{bmatrix} \lambda : \lambda \geq 0 \right\}$
- Otherwise, proceed to the next step

5. **Pivot:** x_k enters the basis and x_{B_r} leaves, where

$$r = \underset{i \in \{1,\dots,m\}}{\operatorname{argmin}} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\} \qquad \text{(minimum ratio test)}$$

- Refresh the basis $\boldsymbol{B}$ (swap $\boldsymbol{a_{B_r}}$ to $\boldsymbol{a_k}$), N, $\boldsymbol{c_B}^T$ and $\boldsymbol{c_N}^T$, and go to the first step

The Simplex Method: Complexity

- **Theorem:** if the simplex method does not encounter a degenerate basis then it solves the linear program in a finite number of steps or proves that the optimal solution is unbounded
- In each iteration we either terminate or find a new basic feasible solution different from the current one
- The number of basic feasible solutions is finite $\square$
- Note: the basis is degenerate if $\boldsymbol{x_B} = \bar{\boldsymbol{b}} \not> \mathbf{0}$
- The objective function value remains the same during the pivot (z_0): we stay in the same extreme point
- **Cycling:** jumping from one degenerate basic feasible solution to the other the simplex stays indefinitely in the same extreme point without improving the objective function
- Finite termination is not guaranteed in such cases: rarely occurs in practice

The Simplex Method: Complexity

- Choosing the entering variable in a different way can prevent cycling (e.g., Bland's pivoting rule)
- But the running time of the simplex method may be exponential in the size of the linear program
- In the worst-case the algorithm may visit each of the $\binom{n}{m}$ basic feasible solutions
- In practice, however, the simplex method is very fast: usually the number of pivots it performs until optimality is linear in m and n
- There exist provably polynomial time algorithms to solve linear programs: Khachian's Ellipsoid Algorithm, Karmarkar's algorithm
- These are interior point solvers, do not use the simplex

The Simplex Tableau

- The simplex algorithm in requires solving three systems of linear equations in each iteration: simple for a computer but difficult for a human
- This can be avoided by using the simplex tableau
- Suppose that we have an initial basis $\boldsymbol{B}$
- Let z be a new variable that specifies the current value of the objective function:

$$z = \boldsymbol{c_B}^T \boldsymbol{x_B} + \boldsymbol{c_N}^T \boldsymbol{x_N}$$

- The linear program augmented with the new variable in tableau form (“tableau”: “tabular representation”, French)

	z	$\boldsymbol{x_B}$	$\boldsymbol{x_N}$	RHS	
z	1	$\boldsymbol{0}$	$\boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c_N}^T$	$\boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{b}$	row 0
$\boldsymbol{x_B}$	$\boldsymbol{0}$	$\boldsymbol{I}_m$	$\boldsymbol{B}^{-1} \boldsymbol{N}$	$\boldsymbol{B}^{-1} \boldsymbol{b}$	rows 1...m

The Simplex Tableau

	z	x_{B_1}	$\dots$	x_{B_m}	x_{N_1}	$\dots$	$x_{N_{n-m}}$	RHS
z	1	0	$\dots$	0	z_1	$\dots$	z_{n-m}	z_0
x_{B_1}	0	1	$\dots$	0	$y_{1,1}$	$\dots$	$y_{1,n-m}$	$\bar{b}_1$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
x_{B_m}	0	0	$\dots$	1	$y_{m,1}$	$\dots$	$y_{m,n-m}$	$\bar{b}_m$

$x_{B_1}, x_{B_2}, \dots, x_{B_m}$: basic variables

$x_{N_1}, x_{N_2}, \dots, x_{N_{n-m}}$: nonbasic variables

z_j: the component of $\boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{N} - \boldsymbol{c_N}^T$ that belongs to the nonbasic variable j and $z_0 = \boldsymbol{c_B}^T \boldsymbol{B}^{-1} \boldsymbol{b}$

$\bar{b}_i$: the i-th element of $\bar{\boldsymbol{b}} = \boldsymbol{B}^{-1} \boldsymbol{b}$

y_{ij}: element at the position (i, j) of matrix $\boldsymbol{B}^{-1} \boldsymbol{N}$

The Simplex Tableau: Pivot

- The objective function has changed: $z + \sum_{j \in N} z_j x_j = z_0$
- The law for choosing the entering variable also changes: x_k enters the basis if $k = \operatorname*{argmin}_{j \in N} z_j$
- The law for choosing the leaving variable remains the same

$$r = \operatorname*{argmin}_{i \in \{1,\ldots,m\}} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$$

- Pivot: using elementary row transformations
  1. Divide row r by y_{rk}
  2. For each $i = 1, 2, \ldots, m : i \neq r$, subtract from row i the new row r multiplied by y_{ik}
  3. Subtract row r multiplied by z_k from the objective row

The Simplex Tableau: Example

- Consider the below linear program

$$\begin{array}{lrcrcrcl} \max & -x_1 & - & x_2 & + & 4x_3 & & \\ \text{s.t.} & x_1 & + & x_2 & + & 2x_3 & \leq & 9 \\ & x_1 & + & x_2 & - & x_3 & \leq & 2 \\ & -x_1 & + & x_2 & + & x_3 & \leq & 4 \\ & x_1, & & x_2, & & x_3 & \geq & 0 \end{array}$$

- Convert to standard for by introducing slack variables:

$$\begin{array}{lrcrcrcrcrcrcl} \max & -x_1 & - & x_2 & + & 4x_3 & + & 0x_4 & + & 0x_5 & + & 0x_6 & & \\ \text{s.t.} & x_1 & + & x_2 & + & 2x_3 & + & x_4 & & & & & = & 9 \\ & x_1 & + & x_2 & - & x_3 & & & + & x_5 & & & = & 2 \\ & -x_1 & + & x_2 & + & x_3 & & & & & + & x_6 & = & 4 \\ & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0 \end{array}$$

The Simplex Tableau: Example

- First we need to find an initial basis: this usually needs some work, but this time we can use a simple trick
- If a linear program is given in canonical form: $\max\{\boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$
- In standard form (the simplex algorithm needs the standard form!): $\max\{\boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} + \boldsymbol{I}\boldsymbol{x}_s = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{x}_s \geq \boldsymbol{0}\}$
- Observe that the columns of the constraint matrix corresponding to the slack variables form an identity matrix: so let $\boldsymbol{B} = \boldsymbol{I}$ (always nonsingular)
- The columns for the slack variables $\boldsymbol{x}_s$ comprise a basis!
- If in addition $\boldsymbol{b} \geq \boldsymbol{0}$, then this basis is also feasible, since then $\bar{\boldsymbol{b}} = \boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{b} \geq \boldsymbol{0}$
- Row 0: since the objective coefficients for the slacks is zero: ${\boldsymbol{c}_{\boldsymbol{B}}}^T\boldsymbol{B}^{-1}\boldsymbol{N} - {\boldsymbol{c}_{\boldsymbol{N}}}^T = -{\boldsymbol{c}_{\boldsymbol{N}}}^T$ and $z_0 = {\boldsymbol{c}_{\boldsymbol{B}}}^T\boldsymbol{B}^{-1}\boldsymbol{b} = 0$

The Simplex Tableau: Example

- We can write the linear program straight into a simplex table (WARNING: row zero must be inverted!)

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	1	1	-4	0	0	0	0
x_4	0	1	1	2	1	0	0	9
x_5	0	1	1	-1	0	1	0	2
x_6	0	-1	1	$\boxed{1}$	0	0	1	4

- The current basis is not optimal since $z_3 = -4$
- The entering variable is x_3, as $z_3 = \min_{j \in N} z_j = -4$
- No unboundedness as $\boldsymbol{y}_3$ is not negative: $y_{i3} > 0$
- The leaving variable is x_6, since $\frac{\bar{b}_6}{y_{63}} = \min\{\frac{9}{2}, 4\} = 4$

The Simplex Tableau: Example

- Perform a pivot by the above rules

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	–3	5	0	0	0	4	16
x_4	0	$\boxed{3}$	–1	0	1	0	–2	1
x_5	0	0	2	0	0	1	1	6
x_3	0	–1	1	1	0	0	1	4

- The new basis is not optimal as $z_1 = -3$
- Thus x_1 enters the basis
- No unboundedness because $y_{41} > 0$, x_4 leaves the basis

The Simplex Tableau: Example

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	4	0	1	0	2	17
x_1	0	1	$-\frac{1}{3}$	0	$\frac{1}{3}$	0	$-\frac{2}{3}$	$\frac{1}{3}$
x_5	0	0	2	0	0	1	1	6
x_3	0	0	$\frac{2}{3}$	1	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{13}{3}$

- The new basis is optimal
- The objective function value can be read from the last element of row 0: $z = 17$
- The basic variables from the RHS column: $\begin{bmatrix} x_1 \\ x_5 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 6 \\ \frac{13}{3} \end{bmatrix}$
- The optimal solution: $\boldsymbol{x}^T = [\frac{1}{3} \quad 0 \quad \frac{13}{3}]$ (note the indices!)