The Simplex Tableau

- Termination: optimality and unbounded optimal solutions
- The steps of the simplex method
- Degeneration and cycling
- Complexity (worst-case and practical)
- The simplex tableau
- Solving linear programs using the simplex tableau: examples

Recall: The Simplex Method

- Let $\boldsymbol{A}$ be an $m \times n$ matrix with $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}, \boldsymbol{b}) = m$, $\boldsymbol{b}$ be a column m-vector, $\boldsymbol{x}$ be a column n-vector, and $\boldsymbol{c}^T$ be a row n-vector, and consider the linear program

$$\begin{aligned} z = \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- Let $\boldsymbol{B}$ be a basis and reorder the columns of $\boldsymbol{A}$ to obtain $\boldsymbol{A} = [\boldsymbol{B} \quad \boldsymbol{N}]$
- Furthermore, let $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_{\boldsymbol{B}} \\ \boldsymbol{x}_{\boldsymbol{N}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}$ the basic solution generated by $\boldsymbol{B}$ and suppose that this basic solution is feasible ($\boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}$)

Recall: The Simplex Method

- The linear program in the nonbasic variable space:

$$\begin{array}{rl} \max & z_0 + \sum_{j \in N} z_j x_j \\ \text{s.t.} & \boldsymbol{x_B} = \bar{\boldsymbol{b}} - \sum_{j \in N} \boldsymbol{y}_j x_j \\ & \boldsymbol{x_B}, \boldsymbol{x_N} \geq \boldsymbol{0} \end{array}$$

  where
  - N denotes the set of nonbasic variables
  - $\bar{\boldsymbol{b}} = \boldsymbol{B}^{-1}\boldsymbol{b}$
  - $\boldsymbol{y}_j$ denotes the column of the matrix $\boldsymbol{B}^{-1}\boldsymbol{N}$ that belongs to the j-th nonbasic variable: $\boldsymbol{y}_j = \boldsymbol{B}^{-1}\boldsymbol{a}_j$
  - $z_0 = \boldsymbol{c_B}^T \mathrm{B}^{-1}\boldsymbol{b} = \boldsymbol{c_B}^T \bar{\boldsymbol{b}}$
  - z_j is the component of the row vector $\boldsymbol{c_N}^T - \boldsymbol{c_B}^T \boldsymbol{B}^{-1}\boldsymbol{N}$ that belongs to the j-th nonbasic variable

Recall: The Simplex Method

- Pivot rules
  - x_k can enter the basis if $z_k > 0$
  - x_r leaves the basis where $r = \operatorname*{argmin}_{i \in \{1,\ldots,m\}} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$
- The optimality condition of the (primal) simplex method: the basic feasible solution $\bar{\boldsymbol{x}}$ is optimal if

$$\forall j \in N : z_j \leq 0$$

- In the below we assume that basic feasible solutions are nondegenerate, i.e., $\bar{\boldsymbol{b}} > \mathbf{0}$
- We handle the degenerate case later

Termination with Unboundedness

- Given a basic feasible solution $\boldsymbol{x}$, let x_k be a nonbasic variable so that $z_k > 0$

$$
\begin{aligned}
\max \quad & z_0 + z_k x_k \\
\text{s.t.} \quad & \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x_B} \\ \boldsymbol{x_N} \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{b}} \\ \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} -\boldsymbol{y}_k \\ \boldsymbol{e_k} \end{bmatrix} x_k \\
& \boldsymbol{x} \geq \boldsymbol{0}
\end{aligned}
$$

- Increasing x_k increases the objective function (by $z_k > 0$)
- We can keep on increasing x_k until some basic variable drops to zero:

$$
x_k \leq \min_{i \in B} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}
$$

Termination with Unboundedness

- This test is well-defined only in the case when there is basic variable i for which $y_{ik} > 0$
- If no such basic variable exists, that is, if $\boldsymbol{y}_k \leq \mathbf{0}$, then no basic variable blocks the growth of x_k
- **Theorem:** the optimal solution of the linear program $\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0}\}$ is **unbounded** if there is basic feasible solution $\bar{\boldsymbol{x}}$ and nonbasic variable x_k so that $z_k > 0$ and $\boldsymbol{y}_k \leq 0$
- **Proof:** $\boldsymbol{d} = \begin{bmatrix} -\boldsymbol{y}_k \\ \boldsymbol{e}_k \end{bmatrix}$ is a **direction** of the feasible region so that all points along the **ray** $\boldsymbol{x} = \bar{\boldsymbol{x}} + \boldsymbol{d}\lambda, \lambda \geq 0$ are feasible
- In this case, for an constant $K > 0$ there is $\lambda > 0$ so that

$$\boldsymbol{c}^T\boldsymbol{x} = z_0 + \lambda z_k > K \qquad \square$$

Unbounded Optimal Solution: Example

- Given the linear program in the canonical form

$$
\begin{array}{lrcrcl}
\max & x_1 & + & 3x_2 & & \\
\text{s.t.} & x_1 & - & 2x_2 & \leq & 4 \\
 & -x_1 & + & x_2 & \leq & 3 \\
 & x_1, & & x_2 & \geq & 0
\end{array}
$$

- Introducing slack variables x_3 and x_4:

$$
\boldsymbol{A} = \begin{bmatrix} 1 & -2 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix}, \boldsymbol{b} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}, \boldsymbol{c}^T = \begin{bmatrix} 1 & 3 & 0 & 0 \end{bmatrix}
$$

- Consider the basis matrix $\boldsymbol{B} = [\boldsymbol{a}_2 \quad \boldsymbol{a}_3] = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}$

$$
\boldsymbol{x_B} = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \end{bmatrix}, \boldsymbol{x_N} = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
$$

Unbounded Optimal Solution: Example

- The usual parameters to transition to the nonbasic space:

$$z_0 = {\boldsymbol{c}_{\boldsymbol{B}}}^T \bar{\boldsymbol{b}} = 9$$

$${\boldsymbol{c}_{\boldsymbol{N}}}^T - {\boldsymbol{c}_{\boldsymbol{B}}}^T \boldsymbol{B}^{-1}\boldsymbol{N} = [1 \quad 0] - [3 \quad 0] \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix} = [4 \quad -3]$$

$$\boldsymbol{B}^{-1}\boldsymbol{N} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 2 \end{bmatrix}$$

- The linear program in the nonbasic variable space:

$$\begin{array}{ll} \max & 9 + 4x_1 - 3x_4 \\ \text{s.t.} & \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \end{bmatrix} x_1 - \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_4 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

Unbounded Optimal Solution: Example

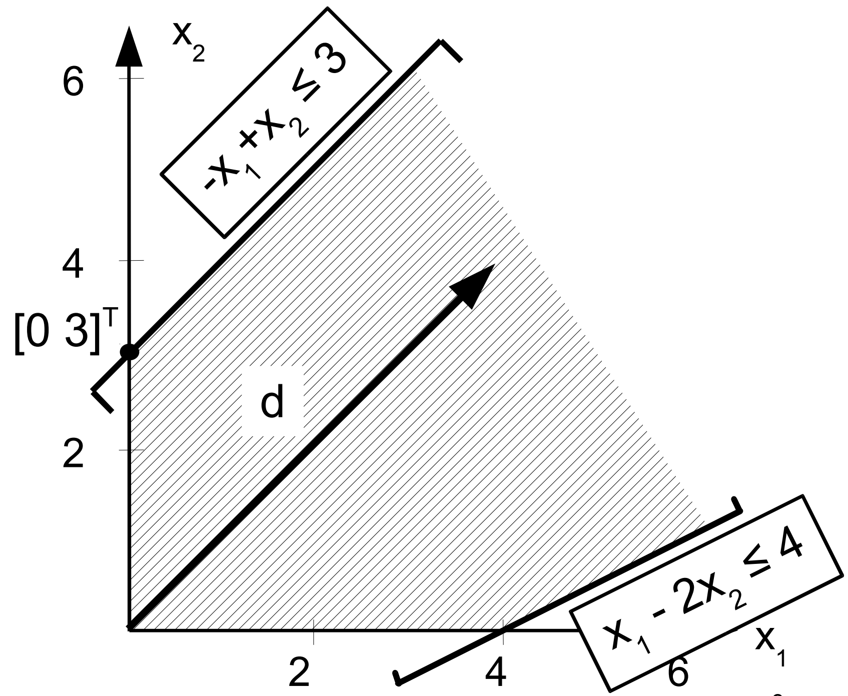
- Observing that $\boldsymbol{y}_1 < \mathbf{0}$, the ray causing unboundedness:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 10 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \lambda, \quad \lambda \geq 0$$

- In the space of the original variables (omitting slacks):

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \lambda : \lambda > 0$$

- Meanwhile, the objective function grows without limit according to $9 + 4\lambda$



The Simplex Method: Initialization

- Let $\boldsymbol{A}$ be an $m \times n$ matrix with $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}, \boldsymbol{b}) = m$, $\boldsymbol{b}$ be a column m-vector, $\boldsymbol{x}$ be a column n-vector, and $\boldsymbol{c}^T$ be a row n-vector, and consider the linear program

$$\begin{aligned} z = \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- Suppose that all basic feasible solutions are nondegenerate
- The simplex method is an iterative algorithm to solve the above linear program, which uses nothing else than a subroutine to solve systems of linear equations and basic linear algebra operations
- **Initialization:** find an initial basic feasible solution and the corresponding basis $\boldsymbol{B}$ (see later on how to do this)

The Simplex Method: Main Step

1. Solve the system $\boldsymbol{B}\boldsymbol{x_B} = \boldsymbol{b}$

- The solution is unique: $\boldsymbol{x_B} = \boldsymbol{B}^{-1}\boldsymbol{b} = \bar{\boldsymbol{b}}$. Let $\boldsymbol{x_N} = \boldsymbol{0}$

2. Solve the system $\boldsymbol{w}^T\boldsymbol{B} = \boldsymbol{c_B}^T$

- The solution is unique: $\boldsymbol{w}^T = \boldsymbol{c_B}^T\boldsymbol{B}^{-1}$
- For each nonbasic variable j obtain the **reduced cost** $z_j = c_j - \boldsymbol{w}^T\boldsymbol{a_j}$ and choose the entering variable as

$$k = \operatorname*{argmax}_{j \in N} z_j$$

(Dantzig's pivot rule)

3. If $z_k \leq 0$ then terminate: $\begin{bmatrix} \boldsymbol{x_B} \\ \boldsymbol{x_N} \end{bmatrix}$ is an optimal solution and the optimal objective function value is $\boldsymbol{c_B}^T\boldsymbol{x_B}$

- Otherwise proceed to the next step

The Simplex Method: Main Step

4. Solve the system $\boldsymbol{B}\boldsymbol{y}_k = \boldsymbol{a_k}$

- The solution is unique: $\boldsymbol{y}_k = \boldsymbol{B}^{-1}\boldsymbol{a_k}$
- If $\boldsymbol{y}_k \leq \mathbf{0}$ then terminate: the linear program is unbounded along the ray $\left\{ \begin{bmatrix} \bar{\boldsymbol{b}} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} -\boldsymbol{y}_k \\ \boldsymbol{e_k} \end{bmatrix} \lambda : \lambda \geq 0 \right\}$
- Otherwise, proceed to the next step

5. **Pivot:** x_k enters the basis and x_{B_r} leaves, where

$$r = \underset{i \in \{1,\ldots,m\}}{\operatorname{argmin}} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\} \qquad \text{(minimum ratio test)}$$

- Refresh the basis $\boldsymbol{B}$ (swap $\boldsymbol{a_{B_r}}$ to $\boldsymbol{a_k}$), N, $\boldsymbol{c_B}^T$ and $\boldsymbol{c_N}^T$, and go to the first step

The Simplex Method: Complexity

- **Theorem:** if the simplex method does not encounter a degenerate basis then it solves the linear program in a finite number of steps or proves that the optimal solution is unbounded
- **Proof:** in each step one of the below 3 cases can occur:
  - if $z_k \leq 0$ then terminate with optimality
  - if $z_k > 0$ but $\boldsymbol{y}_k \leq \mathbf{0}$ then terminate with unboundedness
  - if $z_k > 0$ and $\boldsymbol{y}_k \not\leq \mathbf{0}$ then x_k enters and some x_r leaves the basis, so that $\frac{\bar{b}_r}{y_{rk}} > 0$ and the objective function value strictly increases to $z_k \frac{\bar{b}_r}{y_{rk}} > 0$
- Thus, in each iteration we either terminate or find a new basic feasible solution different from the current one
- The number of basic feasible solutions is finite $\square$

Degeneration and Cycling

- If x_k enters and x_{B_r} leaves the basis and $\boldsymbol{y}_k \not\leq \mathbf{0}$:

Before the pivot	After the pivot
z_0	$z_0 + z_k \dfrac{\bar{b}_r}{y_{rk}}$
$\boldsymbol{x_B}$	$\boldsymbol{x_B} - \dfrac{\bar{b}_r}{y_{rk}} \boldsymbol{y}_k$
$x_k = 0$	$x_k = \dfrac{\bar{b}_r}{y_{rk}}$

- The basis is degenerate if $\boldsymbol{x_B} = \bar{\boldsymbol{b}} \not> \mathbf{0}$
- In this case there is $x_{B_r} = \bar{b}_r = 0$, and if in addition $y_{rk} > 0$ then x_{B_r} may x_k: $\min\limits_{i \in \{1,\ldots,m\}} \left\{ \dfrac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\} = \frac{0}{y_{rk}}$
- The objective function value remains the same (z_0): we stay in the same extreme point during the pivot

Degeneration and Cycling

- **Cycling:** jumping from one degenerate basic feasible solution to the other the simplex stays indefinitely in the same extreme point without improving the objective function
- Finite termination is not guaranteed in such cases
- Cycling is theoretically possible but rarely occurs in practice
- As we may choose the entering variable x_k arbitrarily as long as $z_k > 0$, we can define different **pivot rules**:
  - Dantzig's pivot rule: $k = \mathrm{argmax}_{j \in N} z_j$
  - Greedy: choose the variable that yields the largest gain in the objective function
  - Bland's pivoting rule: choose the smallest index nonbasic variable for which $z_j > 0$ and if $x_{B_r} = 0$ holds for more than one basic variable choose the one with the smallest index as the leaving variable
  - **Theorem:** Bland's pivoting rule prevents cycling

The Simplex Method: Complexity

- The running time of the simplex method may be exponential in the size of the linear program using Dantzig's pivot rule
- In the worst-case the algorithm may visit each of the $\binom{n}{m}$ basic feasible solutions
- In fact, for any deterministic pivot rule an example has been found that yields exponential running time for the simplex
- Open question: is there a deterministic pivot rule that gives a running time better than exponential?
- In practice, however, the simplex method is very fast: usually the number of pivots it performs until optimality is linear in m and n
- There exist provably polynomial time algorithms to solve linear programs: Khachian's Ellipsoid Algorithm, Karmarkar's algorithm

The Simplex Tableau

- The simplex algorithm in requires solving three systems of linear equations in each iteration: simple for a computer but difficult for a human
- This can be avoided by using the simplex tableau
- Suppose that we have an initial basis $\boldsymbol{B}$
- Let z be a new variable that specifies the current value of the objective function:

$$z = \boldsymbol{c_B}^T \boldsymbol{x_B} + \boldsymbol{c_N}^T \boldsymbol{x_N}$$

- The linear program augmented with the new variable:

$$\begin{array}{lcccccc} \max & z & & & & & \\ \text{s.t.} & z & - & \boldsymbol{c_B}^T \boldsymbol{x_B} & - & \boldsymbol{c_N}^T \boldsymbol{x_N} & = & 0 \\ & & & \boldsymbol{B x_B} & + & \boldsymbol{N x_N} & = & \boldsymbol{b} \\ & & & \boldsymbol{x_B}, & & \boldsymbol{x_N} & \geq & \boldsymbol{0} \end{array}$$

The Simplex Tableau

- $\boldsymbol{x_N}$ determines $\boldsymbol{x_B}$ as $\boldsymbol{x_B} + \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x_N} = \boldsymbol{B}^{-1}\boldsymbol{b}$
- Plus we know that $z = \boldsymbol{c_B}^T \mathrm{B}^{-1}\boldsymbol{b} + (\boldsymbol{c_N}^T - \boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{N})\boldsymbol{x_N}$

$$z + \boldsymbol{0}\boldsymbol{x_B} + (\boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c_N}^T)\boldsymbol{x_N} = \boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{b}$$

- In the nonbasic variable space:

$$\begin{array}{lcccccc} \max & z & & & & & \\ \text{s.t.} & z & + & \boldsymbol{0}\boldsymbol{x_B} & + & (\boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c_N}^T)\boldsymbol{x_N} & = & \boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{b} \\ & & & \boldsymbol{I}_m\boldsymbol{x_B} & + & \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x_N} & = & \boldsymbol{B}^{-1}\boldsymbol{b} \\ & & & \boldsymbol{x_B}, & & \boldsymbol{x_N} & \geq & \boldsymbol{0} \end{array}$$

- In tableau form (“tableau”: “tabular representation”, French)

	z	$\boldsymbol{x_B}$	$\boldsymbol{x_N}$	RHS	
z	1	$\boldsymbol{0}$	$\boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c_N}^T$	$\boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{b}$	row 0
$\boldsymbol{x_B}$	$\boldsymbol{0}$	$\boldsymbol{I}_m$	$\boldsymbol{B}^{-1}\boldsymbol{N}$	$\boldsymbol{B}^{-1}\boldsymbol{b}$	rows 1...m

The Simplex Tableau

	z	x_{B_1}	$\ldots$	x_{B_m}	x_{N_1}	$\ldots$	$x_{N_{n-m}}$	RHS
z	1	0	$\ldots$	0	z_1	$\ldots$	z_{n-m}	z_0
x_{B_1}	0	1	$\ldots$	0	$y_{1,1}$	$\ldots$	$y_{1,n-m}$	$\bar{b}_1$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
x_{B_m}	0	0	$\ldots$	1	$y_{m,1}$	$\ldots$	$y_{m,n-m}$	$\bar{b}_m$

$x_{B_1}, x_{B_2}, \ldots, x_{B_m}$: basic variables

$x_{N_1}, x_{N_2}, \ldots, x_{N_{n-m}}$: nonbasic variables

z_j: the component of $\boldsymbol{c_B}^T \boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c_N}^T$ that belongs to the nonbasic variable j and $z_0 = \boldsymbol{c_B}^T \boldsymbol{B}^{-1}\boldsymbol{b}$

$\bar{b}_i$: the i-th element of $\bar{\boldsymbol{b}} = \boldsymbol{B}^{-1}\boldsymbol{b}$

y_{ij}: element at the position (i, j) of matrix $\boldsymbol{B}^{-1}\boldsymbol{N}$

The Simplex Tableau: Entering Variable

- Unfortunately, the objective function has changed:

$$z + \sum_{j \in N} z_j x_j = z_0$$

- To improve the objective function value we need to find a nonbasic variable k for which $z_k < 0$
- The entering variable can be read from row 0
  - Nonbasic variable k enters the basis for which

$$k = \operatorname*{argmin}_{j \in N} z_j$$

  - Optimality condition:

$$z_k = \min_{j \in N} z_j \geq 0$$

The Simplex Tableau: Leaving Variable

- The basic variables according to the simplex tableau:

$$\boldsymbol{I}_m \boldsymbol{x}_{\boldsymbol{B}} + \boldsymbol{B}^{-1} \boldsymbol{N} \boldsymbol{x}_{\boldsymbol{N}} = \boldsymbol{B}^{-1} \boldsymbol{b}$$

- The current value of the basic variable x_{B_i} can be read from row i of the tableau:

$$x_{B_i} + \sum_{j \in N} y_{ij} x_j = \bar{b}_i$$

  - when x_k is increased, x_{B_i} decreases by $y_{ik} x_k$
  - we have unboundedness if no positive element occurs in column k: $\boldsymbol{y}_k \leq 0$
  - the leaving variable: $r = \underset{i \in \{1,\ldots,m\}}{\operatorname{argmin}} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$

The Simplex Tableau: Pivot

- Suppose k enters and r leaves the basis
- The simplex tableau before the pivot:

	z	$x_{B_1} \ldots x_{B_r} \ldots x_{B_m}$	$\ldots x_{N_j} \ldots x_{N_k} \ldots$	RHS
z	1	$0 \quad \ldots \quad 0 \quad \ldots \quad 0$	$\ldots \quad z_j \quad \ldots \quad z_k \quad \ldots$	z_0
x_{B_1}	0	$1 \quad \ldots \quad 0 \quad \ldots \quad 0$	$\ldots \quad y_{1j} \quad \ldots \quad y_{1k} \quad \ldots$	$\bar{b}_1$
$\vdots$	$\vdots$	$\vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$
x_{B_r}	0	$0 \quad \ldots \quad 1 \quad \ldots \quad 0$	$\ldots \quad y_{rj} \quad \ldots \quad y_{rk} \quad \ldots$	$\bar{b}_r$
$\vdots$	$\vdots$	$\vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$
x_{B_m}	0	$0 \quad \ldots \quad 0 \quad \ldots \quad 1$	$\ldots y_{mj} \ldots y_{mk} \ldots$	$\bar{b}_m$

The Simplex Tableau: Pivot

1. Divide row r by y_{rk}

	z	$x_{B_1} \ldots x_{B_r} \ldots x_{B_m}$	$\ldots x_{N_j} \ldots x_{N_k} \ldots$	RHS
z	1	$0 \quad \ldots \quad 0 \quad \ldots \quad 0$	$\ldots \quad z_j \quad \ldots \quad z_k \quad \ldots$	z_0
x_{B_1}	0	$1 \quad \ldots \quad 0 \quad \ldots \quad 0$	$\ldots \; y_{1j} \; \ldots \; y_{1k} \; \ldots$	$\bar{b}_1$
$\vdots$	$\vdots$	$\vdots \qquad \vdots \qquad \vdots$	$\vdots \qquad \vdots$	$\vdots$
x_{B_r}	0	$0 \quad \ldots \quad \frac{1}{y_{rk}} \quad \ldots \quad 0$	$\ldots \; \frac{y_{rj}}{y_{rk}} \; \ldots \quad 1 \quad \ldots$	$\frac{\bar{b}_r}{y_{rk}}$
$\vdots$	$\vdots$	$\vdots \qquad \vdots \qquad \vdots$	$\vdots \qquad \vdots$	$\vdots$
x_{B_m}	0	$0 \quad \ldots \quad 0 \quad \ldots \quad 1$	$\ldots y_{mj} \ldots y_{mk} \ldots$	$\bar{b}_m$

The Simplex Tableau: Pivot

2. For each $i = 1, 2, \ldots, m : i \neq r$, subtract from row i the new row r multiplied by y_{ik}

	z	$x_{B_1} \ldots x_{B_r} \ldots x_{B_m}$	$\ldots \quad x_{N_j} \quad \ldots x_{N_k} \ldots$	RHS
z	1	$0 \ldots 0 \ldots 0$	$\ldots \quad z_j \quad \ldots z_k \ldots$	z_0
x_{B_1}	0	$1 \ldots \frac{-y_{1k}}{y_{rk}} \ldots 0$	$\ldots \; y_{1j} - y_{1k}\frac{y_{rj}}{y_{rk}} \; \ldots \; 0 \; \ldots$	$\bar{b}_1 - y_{1k}\frac{\bar{b}_r}{y_{rk}}$
$\vdots$	$\vdots$	$\vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$
x_{B_r}	0	$0 \ldots \frac{1}{y_{rk}} \ldots 0$	$\ldots \quad \frac{y_{rj}}{y_{rk}} \quad \ldots \; 1 \; \ldots$	$\frac{\bar{b}_r}{y_{rk}}$
$\vdots$	$\vdots$	$\vdots \quad \vdots \quad \vdots$	$\vdots \quad \vdots$	$\vdots$
x_{B_m}	0	$0 \ldots \frac{-y_{mk}}{y_{rk}} \ldots 1$	$\ldots \; y_{mj} - y_{mk}\frac{y_{rj}}{y_{rk}} \; \ldots \; 0 \; \ldots$	$\bar{b}_m - y_{mk}\frac{\bar{b}_r}{y_{rk}}$

The Simplex Tableau: Pivot

3. Subtract from the objective row the new row r multiplied by z_k

	z	$x_{B_1} \ldots \quad x_{B_r} \quad \ldots x_{B_m}$	$\ldots \quad x_{N_j} \quad \ldots x_{N_k} \ldots$	RHS
z	1	$0 \quad \ldots -z_k \frac{y_{1k}}{y_{rk}} \ldots \quad 0$	$\ldots \quad z_j - z_k \frac{y_{rj}}{y_{rk}} \quad \ldots \quad 0 \quad \ldots$	$z_0 - z_k \frac{\bar{b}_r}{y_{rk}}$
x_{B_1}	0	$1 \quad \ldots \quad \frac{-y_{1k}}{y_{rk}} \quad \ldots \quad 0$	$\ldots \; y_{1j} - y_{1k} \frac{y_{rj}}{y_{rk}} \quad \ldots \quad 0 \quad \ldots$	$\bar{b}_1 - y_{1k} \frac{\bar{b}_r}{y_{rk}}$
$\vdots$	$\vdots$	$\vdots \qquad \vdots \qquad \vdots$	$\vdots \qquad \vdots$	$\vdots$
x_k	0	$0 \quad \ldots \quad \frac{1}{y_{rk}} \quad \ldots \quad 0$	$\ldots \quad \frac{y_{rj}}{y_{rk}} \quad \ldots \quad 1 \quad \ldots$	$\frac{\bar{b}_r}{y_{rk}}$
$\vdots$	$\vdots$	$\vdots \qquad \vdots \qquad \vdots$	$\vdots \qquad \vdots$	$\vdots$
x_{B_m}	0	$0 \quad \ldots \quad \frac{-y_{mk}}{y_{rk}} \quad \ldots \quad 1$	$\ldots \; y_{mj} - y_{mk} \frac{y_{rj}}{y_{rk}} \; \ldots \quad 0 \quad \ldots$	$\bar{b}_m - y_{mk} \frac{\bar{b}_r}{y_{rk}}$

- x_k has entered the basis so in the tableau obtained row r now corresponds to x_k
- x_{B_r} has left the basis and changed to nonbasic

The Simplex Tableau: Example

- Consider the below linear program

$$\begin{array}{lrcrcrcl} \max & -x_1 & - & x_2 & + & 4x_3 & & \\ \text{s.t.} & x_1 & + & x_2 & + & 2x_3 & \leq & 9 \\ & x_1 & + & x_2 & - & x_3 & \leq & 2 \\ & -x_1 & + & x_2 & + & x_3 & \leq & 4 \\ & x_1, & & x_2, & & x_3 & \geq & 0 \end{array}$$

- Convert to standard for by introducing slack variables:

$$\begin{array}{lrcrcrcrcrcrcl} \max & -x_1 & - & x_2 & + & 4x_3 & + & 0x_4 & + & 0x_5 & + & 0x_6 & & \\ \text{s.t.} & x_1 & + & x_2 & + & 2x_3 & + & x_4 & & & & & = & 9 \\ & x_1 & + & x_2 & - & x_3 & & & + & x_5 & & & = & 2 \\ & -x_1 & + & x_2 & + & x_3 & & & & & + & x_6 & = & 4 \\ & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0 \end{array}$$

The Simplex Tableau: Example

- First we need to find an initial basis: let this be the columns of the slack variables for now
- Given a linear program $\max\{\boldsymbol{c^T x} : \boldsymbol{Ax} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ in canonical form the columns for the slack variables $\boldsymbol{x}_s$ always comprise a basis: $\max\{\boldsymbol{c^T x} : \boldsymbol{Ax} + \boldsymbol{I}\boldsymbol{x}_s = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}, \boldsymbol{x}_s \geq \boldsymbol{0}\}$ and $\boldsymbol{B} = \boldsymbol{I}$ is always nonsingular
- If in addition $\boldsymbol{b} \geq \boldsymbol{0}$, then this basis is also feasible, since then $\bar{\boldsymbol{b}} = \boldsymbol{B}^{-1}\boldsymbol{b} = \boldsymbol{b} \geq \boldsymbol{0}$
- Otherwise, finding an initial basis requires special steps
- So let $\boldsymbol{B} = [\boldsymbol{a_4} \quad \boldsymbol{a_5} \quad \boldsymbol{a_6}]$
- Row 0: $\boldsymbol{c_B}^T \boldsymbol{B}^{-1}\boldsymbol{N} - \boldsymbol{c_N}^T = -\boldsymbol{c_N}^T$ as the objective function coefficients of the basic variables $\boldsymbol{x}_s$ are all zero (by $\boldsymbol{x}_s$ being slacks) and so $\boldsymbol{c_B}^T = \boldsymbol{0}$, and similarly $z_0 = \boldsymbol{c_B}^T \boldsymbol{B}^{-1}\boldsymbol{b} = 0$

The Simplex Tableau: Example

- The initial simplex table (recall: row zero must be inverted!)

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	1	1	-4	0	0	0	0
x_4	0	1	1	2	1	0	0	9
x_5	0	1	1	-1	0	1	0	2
x_6	0	-1	1	$\boxed{1}$	0	0	1	4

- The current basis is not optimal since $z_3 = -4$
- The entering variable is x_3, as $z_3 = \min_{j \in N} z_j = -4$
- No unboundedness as $\boldsymbol{y}_3$ is not negative: $y_{i3} > 0$
- The leaving variable is x_6, since $\frac{\bar{b}_6}{y_{63}} = \min\{\frac{9}{2}, 4\} = 4$

Simplex Pivot: Example

1. Divide the row of x_6 (the last row) by y_{63} (now equals 1)

- Note that the rows of the tableau are identified by the corresponding basic variable and not the row index

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	1	1	-4	0	0	0	0
x_4	0	1	1	2	1	0	0	9
x_5	0	1	1	-1	0	1	0	2
x_6	0	-1	1	$\boxed{1}$	0	0	1	4

Simplex Pivot: Example

2. Subtract from the row of x_4 (x_5) the new row of x_6 multiplied by y_{4k} (y_{5k}, respectively)
   - so subtract from the row of x_4 two times the row of x_6
   - idea is to obtain zero for y_{43} and y_{53} (marked in bold) by elementary row operations

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	1	1	-4	0	0	0	0
x_4	0	3	-1	$\mathbf{0}$	1	0	-2	1
x_5	0	0	2	$\mathbf{0}$	0	1	1	6
x_6	0	-1	1	$\boxed{1}$	0	0	1	4

Simplex Pivot: Example

3. Subtract from row zero the row of x_6 multiplied by z_3
   - so add four times the row of x_6 to row 0
   - again, idea is to zero out z_3 (marked in bold)!

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	-3	5	$\mathbf{0}$	0	0	4	16
x_4	0	$\boxed{3}$	-1	0	1	0	-2	1
x_5	0	0	2	0	0	1	1	6
x_3	0	-1	1	1	0	0	1	4

- Pivot ready, obtained the new basis $B = \{x_3, x_4, x_5\}$
- The last row of the tableau now belongs to the entering variable x_3, always worth marking in the tableau

The Simplex Tableau: Example

- Use of the simplex tableau
  - the last element in row 0 (the objective row) specifies the objective function value in the current basis, now $z = 16$
  - the current values for the basic variables can be read from the RHS column (if some value is negative then there has been a mistake)
  - if no negative values in row 0 then the current basis is optimal

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	-3	5	0	0	0	4	16
x_4	0	3	-1	0	1	0	-2	1
x_5	0	0	2	0	0	1	1	6
x_3	0	-1	1	1	0	0	1	4

The Simplex Tableau: Example

- The current basis is not optimal as $z_1 = -3$
- Thus x_1 enters the basis
- No unboundedness because $y_{41} > 0$, x_4 leaves the basis

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	−3	5	0	0	0	4	16
x_4	0	$\boxed{3}$	−1	0	1	0	−2	1
x_5	0	0	2	0	0	1	1	6
x_3	0	−1	1	1	0	0	1	4

- Divide by 3 the row of x_4, then add the row obtained to the x_3 row in order to zero out y_{31}, and finally add 3 times the objective row to eliminate z_1

The Simplex Tableau: Example

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	4	0	1	0	2	17
x_1	0	1	$-\frac{1}{3}$	0	$\frac{1}{3}$	0	$-\frac{2}{3}$	$\frac{1}{3}$
x_5	0	0	2	0	0	1	1	6
x_3	0	0	$\frac{2}{3}$	1	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{13}{3}$

- The new basis is optimal
- The objective function value can be read from the last element of row 0: $z = 17$
- The basic variables from the RHS column: $\begin{bmatrix} x_1 \\ x_5 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 6 \\ \frac{13}{3} \end{bmatrix}$
- The optimal solution: $\boldsymbol{x}^T = [\frac{1}{3} \quad 0 \quad \frac{13}{3}]$ (note the indices!)

The Simplex Tableau: Example

- Solve the below linear program

$$
\begin{array}{lrcrcrcc}
\max & -2x_1 & + & x_2 & + & 3x_3 & & \\
\text{s.t.} & 2x_1 & - & 3x_2 & + & x_3 & \leq & 0 \\
 & & - & 2x_2 & + & 4x_3 & \leq & 1 \\
 & -x_1 & - & x_2 & & & \leq & 3 \\
 & x_1, & & x_2, & & x_3 & \geq & 0
\end{array}
$$

- Convert to standard for by introducing slack variables:

$$
\begin{array}{lrcrcrcrcrcrcc}
\max & -2x_1 & + & x_2 & + & 3x_3 & + & 0x_4 & + & 0x_5 & + & 0x_6 & & \\
\text{s.t.} & 2x_1 & - & 3x_2 & + & x_3 & + & x_4 & & & & & = & 0 \\
 & & - & 2x_2 & + & 4x_3 & & & + & x_5 & & & = & 1 \\
 & -x_1 & - & x_2 & & & & & & & + & x_6 & = & 3 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6 & \geq & 0
\end{array}
$$

The Simplex Tableau: Example

- Choosing the slacks as the basis, the initial simplex tableau:

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	2	-1	-3	0	0	0	0
x_4	0	2	-3	$\boxed{1}$	1	0	0	0
x_5	0	0	-2	4	0	1	0	1
x_6	0	-1	-1	0	0	0	1	3

- x_3 enters the basis and x_4 leaves
- Subtract 4 times the row of x_4 from x_5's row and add three times it to row 0
- Degenerate pivot since $b_4 = 0$: after the picot we'll remain at the same extreme point

The Simplex Tableau: Example

- The simplex tableau after the pivot:

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	8	-10	0	3	0	0	0
x_3	0	2	-3	1	1	0	0	0
x_5	0	-8	$\boxed{10}$	0	-4	1	0	1
x_6	0	-1	-1	0	0	0	1	3

- x_2 enters and x_5 leaves the basis

The Simplex Tableau: Example

- The new simplex tableau after the pivot:

	z	x_1	x_2	x_3	x_4	x_5	x_6	RHS
z	1	0	0	0	-1	1	0	1
x_3	0	$-\frac{2}{5}$	0	1	$-\frac{1}{5}$	$\frac{3}{10}$	0	$\frac{3}{10}$
x_2	0	$-\frac{4}{5}$	1	0	$-\frac{2}{5}$	$\frac{1}{10}$	0	$\frac{1}{10}$
x_6	0	$-\frac{9}{5}$	0	0	$-\frac{2}{5}$	$\frac{1}{10}$	1	$\frac{31}{10}$

- The entering variable is x_4
- The column $\boldsymbol{y}_4$ for x_4 is all negative: x_4 can be freely increased without any basic variable dropping to zero and blocking x_4, meanwhile the objective function value grows without limit
- Unbounded optimal solution

The Simplex Tableau: Example

- Using the simplex tableau we can also obtain the ray $\bar{\boldsymbol{x}} + \lambda \boldsymbol{d} : \lambda \geq 0$ causing unboundedness
  - $\bar{\boldsymbol{x}}$ is the current basic feasible solution
  - $\boldsymbol{d}$ comes from the column of the simplex tableau that corresponds to variable x_4

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{10} \\ \frac{3}{10} \\ 0 \\ 0 \\ \frac{31}{10} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{2}{5} \\ \frac{1}{5} \\ 1 \\ 0 \\ \frac{2}{5} \end{bmatrix} \lambda, \quad \lambda \geq 0$$

- Take note of the indices and the plus/minus signs!