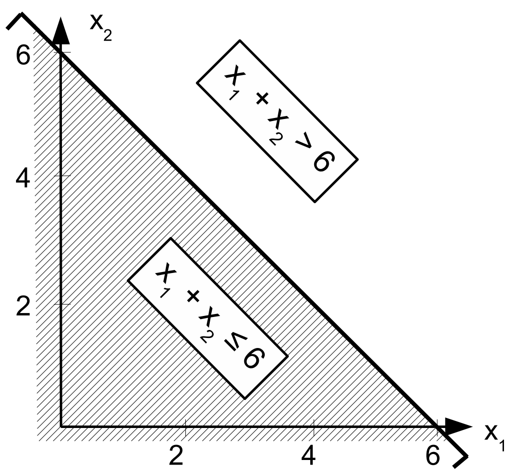
The Simplex Method: A Summary

*WARNING: this is just a summary of the material covered in the full slide-deck **The Simplex Method** that will orient you as per the topics covered there; you are required to learn the full version, not just this summary!*

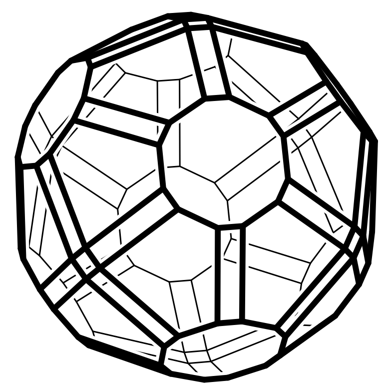
- Recall: linear programs and the two “Fundamental Theorems”
- Basics: basic solutions, basic feasible solutions, and degenerate basic feasible solutions
- Iteration of the simplex method: the initial basic feasible solution, the linear program in the nonbasic variable space, the entering and leaving variables, the pivot
- Termination: termination with optimality and alternative optima

Linear Programs Are Convex Programs

- A linear program asks for the maximum of an **objective function** $\boldsymbol{c}^T\boldsymbol{x}$ over a **feasible region** $\{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\} \subseteq \mathbb{R}^n$
  - the objective is linear: both convex and concave
  - the feasible region is a polyhedron: convex by definition
- Recall, a polyhedron is a geometric object with "flat sides": the intersection of finitely many halfspaces



a half-space



a convex polyhedron

- A linear program is a **convex program**

Linear Programs Are Convex Programs

- **Fundamental Theorem of Convex Programming:** Let X be a nonempty **convex set** in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a **concave function** on X. Then, if some $\bar{\boldsymbol{x}} \in X$ is a local optimum of the optimization problem $\max f(\boldsymbol{x}) : \boldsymbol{x} \in X$ then $\bar{\boldsymbol{x}}$ is also a global optimum
- It is enough to search for a local optimum (much easier)
- The “hill-climbing” algorithm is correct for linear programs:
  - start from a feasible point, try to find a direction moving along which improves the objective function
  - if no such direction exists, point is a local optimum so it is also a global one, terminate with the current point
  - otherwise, move along the direction as long as possible to get a “better” feasible solution
  - repeat the iteration

Linear Programs and Extreme Points

- If the polyhedron of the feasible region X is bounded then X can be written equivalently as a convex combination of its extreme points $\boldsymbol{x}_j$ (Minkowski-Weyl)

$$X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\} = \operatorname{conv}\left(\boldsymbol{x}_j : j \in \{1, \ldots, k\}\right)$$

- Half-space representation $\equiv$ extreme-point representation

- **The Fundamental Theorem of LP:** given a linear program $\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$, if an optimal solution exists then there is at least one optimal solution that occurs at an extreme point of the feasible region

The Simplex Algorithm: Idea

- The simplex algorithm for solving linear programs is a culmination of the above ideas

1. The **simplex algorithm uses the hill-climbing scheme**: move along an improving feasible direction if possible or conclude that the current solution is a global optimum
2. The **simplex algorithm considers only the extreme points of the feasible region**: no need to search in the interior of the feasible set

- As such, the simplex algorithm considers only the finite set of extreme points: terminates in finite steps
- Although not a polynomial-time algorithm, it is still perhaps the most practical choice for solving linear programs

Basic Feasible Solutions

- The notion of extreme points gives a purely geometric interpretation for the optimal solution of a linear program
- In order to define systematic solver algorithm we need an algebraic interpretation: basic feasible solutions
- Consider a linear program whose the feasible region is given by the polyhedron $X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$, where $\boldsymbol{A}$ is an $m \times n$ matrix, $\boldsymbol{b}$ is a column m-vector, and $\boldsymbol{x}$ is a column n-vector
- Suppose that $\mathrm{rank}(\boldsymbol{A}) = \mathrm{rank}(\boldsymbol{A}, \boldsymbol{b}) = m$
- Reordering columns of $\boldsymbol{A}$ so that the first m columns are linearly independent, we write $\boldsymbol{A} = [\boldsymbol{B} \quad \boldsymbol{N}]$ where
  - $\boldsymbol{B}$ is an $m \times m$ quadratic nonsingular matrix, called the **basic matrix**
  - $\boldsymbol{N}$ is an $m \times (n - m)$ matrix, called the **nonbasic** matrix

Basic Feasible Solutions

- Reorder $\boldsymbol{x}$ accordingly: $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{bmatrix}$, where $\boldsymbol{x}_B$ contains the **basic variables** (or dependent variables) that belong to the columns of $\boldsymbol{B}$, and $\boldsymbol{x}_N$ contains the **nonbasic variables** (or independent variables) that belong to the columns of $\boldsymbol{N}$
- An explicit representation of the basic variables in the terms of the nonbasic variables ($\boldsymbol{x}_N$ can be chosen arbitrarily!)

$$\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_N$$

- Choosing $\boldsymbol{x}_N = 0$ gives the **basic solution** (in the basis $\boldsymbol{B}$)

$$\boldsymbol{x}_B = \boldsymbol{B}^{-1}\boldsymbol{b}, \;\; \boldsymbol{x}_N = \boldsymbol{0}, \;\; \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}$$

- If $\boldsymbol{x}_B \geq \boldsymbol{0}$ then $(\boldsymbol{x}_B, \boldsymbol{x}_N)$ is a **basic feasible solution**

Basic Feasible Solutions

- If all components of $\boldsymbol{x}_B$ are strictly positive ($\boldsymbol{x}_B > \mathbf{0}$) then $\boldsymbol{x}$ is a **nondegenerate basic (feasible) solution**, otherwise it is a **degenerate basic (feasible) solution**
- Basic feasible solutions are essential as these give the algebraic interpretation for the geometric notion of extreme points (which we know are key to solving linear programs)
- **Theorem:** given a linear program $\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0}\}$, $\boldsymbol{x}$ is a basic feasible solution if and only if $\boldsymbol{x}$ is an extreme point of the feasible region $X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0}\}$
- Iterating along basic feasible solutions (i.e., extreme points) we can solve a linear program to optimality: the simplex method

The Simplex Method

- Let $\boldsymbol{A}$ be an $m \times n$ matrix with $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}, \boldsymbol{b}) = m$, $\boldsymbol{b}$ be a column m-vector, $\boldsymbol{x}$ be a column n-vector, and $\boldsymbol{c}^T$ be a row n-vector, and consider the linear program

$$\begin{aligned} z = \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- Let $\boldsymbol{B}$ be an initial basis and $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_{\boldsymbol{B}} \\ \boldsymbol{x}_{\boldsymbol{N}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix} \geq \boldsymbol{0}$ be the **basic feasible solution** for $\boldsymbol{B}$ and write $\boldsymbol{A} = [\boldsymbol{B} \quad \boldsymbol{N}]$
- We rewrite the linear program in the space of nonbasic variables

The Simplex Method

- **Theorem:** the **linear program in the nonbasic variable space** is given by

$$\begin{aligned} \max \quad & z_0 + \textstyle\sum_{j \in N} z_j x_j \\ \text{s.t.} \quad & \boldsymbol{x_B} = \bar{\boldsymbol{b}} - \textstyle\sum_{j \in N} \boldsymbol{y}_j x_j \\ & \boldsymbol{x_B}, \boldsymbol{x_N} \geq \mathbf{0} \end{aligned}$$

  where
  - N denotes the set of nonbasic variables
  - $\bar{\boldsymbol{b}} = \boldsymbol{B}^{-1}\boldsymbol{b}$
  - $\boldsymbol{y}_j$ denotes the column of the matrix $\boldsymbol{B}^{-1}\boldsymbol{N}$ that belongs to the j-th nonbasic variable
  - $z_0 = \boldsymbol{c_B}^T \mathrm{B}^{-1}\boldsymbol{b} = \boldsymbol{c_B}^T \bar{\boldsymbol{b}}$
  - z_j is the component of the row vector $\boldsymbol{c_N}^T - \boldsymbol{c_B}^T \boldsymbol{B}^{-1}\boldsymbol{N}$ that belongs to the j-th nonbasic variable

The Simplex Method: Example

- Consider the linear program

$$
\begin{array}{lccccc}
\max & x_1 & + & x_2 & & \\
\text{s.t.} & x_1 & + & 2x_2 & \leq & 4 \\
 & & & x_2 & \leq & 1 \\
 & x_1, & & x_2 & \geq & 0
\end{array}
$$

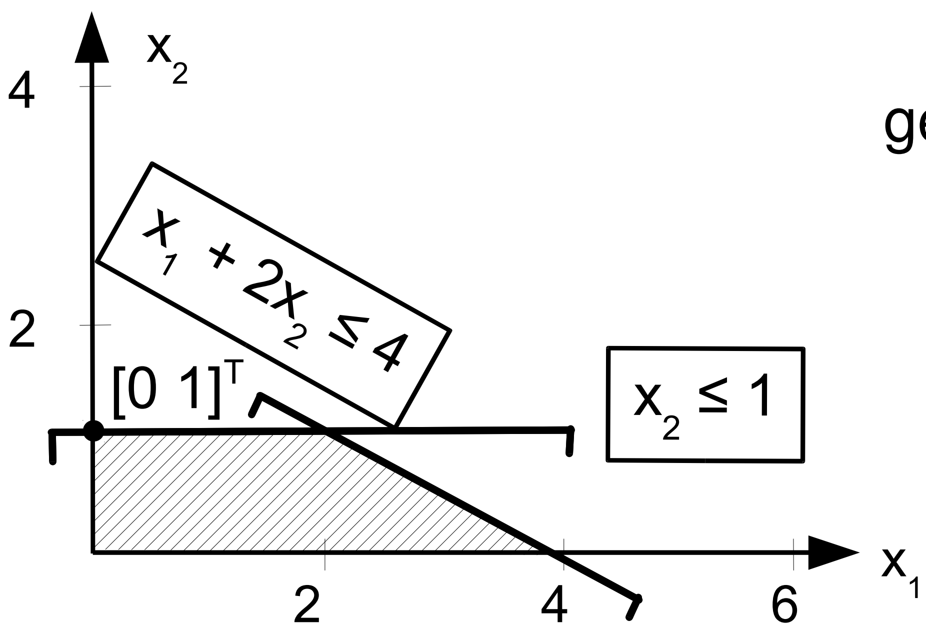
- Introducing slack variables and converting to standard form:

$$
\begin{aligned}
\max \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \boldsymbol{x} & \\
\begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \boldsymbol{x} &= \begin{bmatrix} 4 \\ 1 \end{bmatrix} \\
x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

The Simplex Method: Example

- Converting in the space of nonbasic variables $N = \{1, 4\}$:

$$\begin{aligned} \max \quad & 1 + x_1 - x_4 \\ \text{s.t.} \quad & \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_1 - \begin{bmatrix} 1 \\ -2 \end{bmatrix} x_4 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$



The basic feasible solution generated by the current basis matrix $\boldsymbol{B}$:

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

The Simplex Method: Pivot

- **Theorem:** the optimality condition for the simplex method

$$\forall j \in N : z_j \leq 0$$

- Easily, if for some basic feasible solution $\boldsymbol{x}$ we have $\forall j \in N : z_j \leq 0$, then $\boldsymbol{x}$ is a local optimum, ans so also a global optimum
- Otherwise, let x_k be a nonbasic variable with $z_k > 0$
- Increasing x_k improves the objective function, but this can be done only as long as no basic variable drops to zero
- The first basic variable x_r that drops to zero:

$$r = \operatorname*{argmin}_{i \in \{1,\ldots,m\}} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$$

The Simplex Method: Pivot

- This transformation is called the **pivot**, during which
  - the nonbasic variable x_k with $z_k > 0$ increases from zero and **enters** the basis
  - the basic variable x_r drops to zero and **leaves** the basis

$$r = \operatorname*{argmin}_{i \in \{1,\ldots,m\}} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$$

  - all remaining nonbasic variables remain zero and all remaining basic variables remain nonnegative
- **Theorem:** the pivot results a new basic feasible solution

Unique and Alternative Optima

- **Definition:** the feasible solution $\bar{\boldsymbol{x}}$ to the linear program $\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ is a **unique optimal solution** if for each feasible solution $\boldsymbol{x} \neq \bar{\boldsymbol{x}}$ we have $\boldsymbol{c}^T\boldsymbol{x} < \boldsymbol{c}^T\bar{\boldsymbol{x}}$
- **Theorem:** the basic feasible solution $\bar{\boldsymbol{x}}$ is a unique optimal solution if $\forall j \in N : z_j < 0$
- Easily, as the objective function in the nonbasic space equals $z = z_0 + \sum_{j \in N} z_j x_j$, the condition $\forall j \in N : z_j < 0$ means that increasing *any* x_k will lead strictly worse solutions
- Otherwise, if $\forall j \in N : z_j \leq 0$ holds but there is $x_k : z_k = 0$ then there are alternative optimal solutions