The Simplex Method

- Recall: the Representation Theorem and the Fundamental Theorem
- Basics: basic solutions, basic feasible solutions, and degenerate basic feasible solutions
- Iteration of the simplex method: the initial basic feasible solution, the linear program in the nonbasic variable space, the entering and leaving variables, the pivot
- Termination: termination with optimality

Linear Programs and Extreme Points

- **The Fundamental Theorem:** given a linear program $\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$, if an optimal solution exists then there is at least one optimal solution that occurs at an extreme point of the feasible region
- If the polyhedron of the feasible region X is bounded then X can be written equivalently as a convex combination of its extreme points $\boldsymbol{x}_j$ (Minkowski-Weyl)

$$X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\} = \mathrm{conv}\left(\boldsymbol{x}_j : j \in \{1, \ldots, k\}\right)$$

- **Recall:** the convex combination

$$\mathrm{conv}(\boldsymbol{x}_j) = \left\{\boldsymbol{x} : \boldsymbol{x} = \sum_{j=1}^{k} \lambda_j \boldsymbol{x}_j, \sum_{j=1}^{k} \lambda_j = 1, \lambda_j \geq 0\right\}$$

Linear Programs and Extreme Points

- An equivalent linear program using the coefficients of the convex combination λ_j as variables

$$\max \left\{ \sum_{j=1}^{k} (\boldsymbol{c}^T \boldsymbol{x}_j)\lambda_j : \sum_{j=1}^{k} \lambda_j = 1, \lambda_j \geq 0 \quad \forall j \in \{1, \ldots, k\} \right\}$$

- A solution is guaranteed to occur at an extreme point

$$\boldsymbol{x}_{\mathrm{opt}} = \operatorname*{argmax}_{\boldsymbol{x}_j : j \in \{1,\ldots,k\}} \boldsymbol{c}^T \boldsymbol{x}_j$$

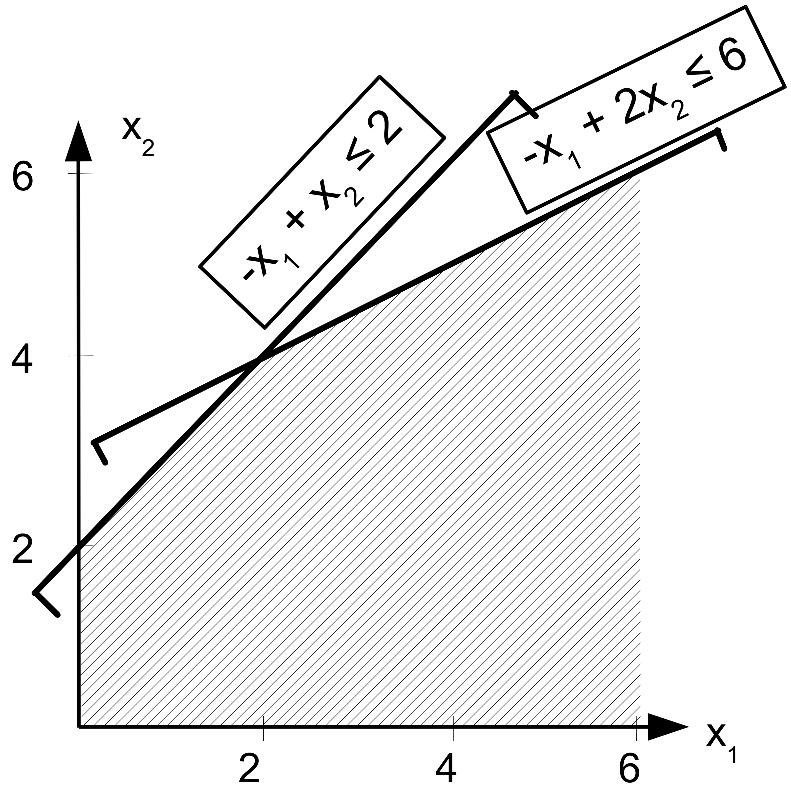
$$z_{\mathrm{opt}} = \max_{j \in \{1,\ldots,k\}} \boldsymbol{c}^T \boldsymbol{x}_j$$

where z_{opt} denotes the optimal objective function value

Extreme Point Solutions: Example

- Consider the below set of constraints

$$\begin{array}{rcrcl} -x_1 & + & x_2 & \leq & 2 \\ -x_1 & + & 2x_2 & \leq & 6 \\ x_1, & & x_2 & \geq & 0 \end{array}$$



- Extreme points:

$$\boldsymbol{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$$\boldsymbol{x}_3 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Extreme Point Solutions: Example

- Let us maximize the objective function $-4x_1 + x_2$

  $\boldsymbol{c}^T\boldsymbol{x}_1 = \begin{bmatrix}-4 & 1\end{bmatrix}\begin{bmatrix}0\\0\end{bmatrix} = 0,\ \boldsymbol{c}^T\boldsymbol{x}_2 = \begin{bmatrix}-4 & 1\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix} = 2,$

  $\boldsymbol{c}^T\boldsymbol{x}_3 = \begin{bmatrix}-4 & 1\end{bmatrix}\begin{bmatrix}2\\4\end{bmatrix} = -4$

- The solution is the extreme point that minimizes the scalar product $\boldsymbol{c}^T\boldsymbol{x}_j$

$$\boldsymbol{x}_{\text{opt}} = \operatorname*{argmax}_{\boldsymbol{x}_j : j\in\{1,\ldots,k\}} \boldsymbol{c}^T\boldsymbol{x}_j = \begin{bmatrix}0\\2\end{bmatrix}$$

$$z_{\text{opt}} = \max_{j\in\{1,\ldots,k\}} \boldsymbol{c}^T\boldsymbol{x}_j = 2$$

Extreme Point Solutions: Example

- If now the task is to maximize the objective function $-x_1 + 3x_2$ then there is no bounded optimal solution
- Since now there is a direction $\boldsymbol{d}$ and a feasible solution $\boldsymbol{x}_0$ so that, starting from the point $\boldsymbol{x}_0$ along the direction $\boldsymbol{d}$ we can obtain arbitrarily large objective function values
- That is, all solutions of the form $\boldsymbol{x}_0 + \mu\boldsymbol{d}$ for any $\mu \geq 0$ are feasible and improve the objective function value
- For example, if $\boldsymbol{d} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\boldsymbol{x}_0 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, then the points along the ray $\boldsymbol{x} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \mu \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mu \geq 0$ are all feasible and the objective function value is $\boldsymbol{c}^T(\boldsymbol{x}_0 + \mu\boldsymbol{d}) = \boldsymbol{c}^T\boldsymbol{x}_0 + \mu\boldsymbol{c}^T\boldsymbol{d}$
- **Lemma:** if $\exists \boldsymbol{d}, \boldsymbol{x}_0$ so that $\boldsymbol{x}_0 + \mu\boldsymbol{d}, \mu \geq 0$ is feasible and $\boldsymbol{c}^T\boldsymbol{d} > 0$ then the optimal solution is unbounded

Basic Feasible Solutions

- The notion of extreme points give a geometric interpretation for the optimal solution of a linear program
- In order to define systematic solver algorithm we need an algebraic interpretation: basic feasible solutions
- Consider a linear program whose the feasible region is given by the polyhedron $X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$, where $\boldsymbol{A}$ is an $m \times n$ matrix, $\boldsymbol{b}$ is a column m-vector, and $\boldsymbol{x}$ is a column n-vector
- Suppose that $\operatorname{rank}(\boldsymbol{A}) = \operatorname{rank}(\boldsymbol{A}, \boldsymbol{b}) = m$
- Reordering columns of $\boldsymbol{A}$ so that the first m columns are linearly independent, we write $\boldsymbol{A} = [\boldsymbol{B} \quad \boldsymbol{N}]$ where
  - $\boldsymbol{B}$ is an $m \times m$ quadratic nonsingular matrix, called the **basic matrix**
  - $\boldsymbol{N}$ is an $m \times (n - m)$ matrix, called the **nonbasic** matrix

Basic Feasible Solutions

- Reorder $\boldsymbol{x}$ accordingly: $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{bmatrix}$, where $\boldsymbol{x}_B$ contains the **basic variables** (or dependent variables) that belong to the columns of $\boldsymbol{B}$, and $\boldsymbol{x}_N$ contains the **nonbasic variables** (or independent variables) that belong to the columns of $\boldsymbol{N}$
- The constraint system in terms of the basis $\boldsymbol{B}$

$$\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{N} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{bmatrix} = \boldsymbol{B}\boldsymbol{x}_B + \boldsymbol{N}\boldsymbol{x}_N = \boldsymbol{b}$$

- Since $\boldsymbol{B}$ is nonsingular, we can multiply with the inverse $\boldsymbol{B}^{-1}$ from the left

$$\boldsymbol{x}_B + \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_N = \boldsymbol{B}^{-1}\boldsymbol{b}$$

Basic Feasible Solutions

- An explicit representation of the basic variables in the terms of the nonbasic variables

$$\boldsymbol{x}_{\boldsymbol{B}} = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_{\boldsymbol{N}}$$

- Note that $\boldsymbol{x}_{\boldsymbol{N}}$ can be chosen arbitrarily!
- Choosing $\boldsymbol{x}_{\boldsymbol{N}} = 0$ gives the **basic solution** (in the basis $\boldsymbol{B}$)

$$\boldsymbol{x}_{\boldsymbol{B}} = \boldsymbol{B}^{-1}\boldsymbol{b}, \;\; \boldsymbol{x}_{\boldsymbol{N}} = \boldsymbol{0}, \;\; \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_{\boldsymbol{B}} \\ \boldsymbol{x}_{\boldsymbol{N}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix}$$

- If in addition $\boldsymbol{x}_{\boldsymbol{B}} \geq \boldsymbol{0}$ then what we have obtained is a **basic feasible solution**

$$\boldsymbol{x}_{\boldsymbol{B}} = \boldsymbol{B}^{-1}\boldsymbol{b} \geq \boldsymbol{0}, \;\; \boldsymbol{x}_{\boldsymbol{N}} = \boldsymbol{0}, \;\; \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_{\boldsymbol{B}} \\ \boldsymbol{x}_{\boldsymbol{N}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix} \geq \boldsymbol{0}$$

Basic Feasible Solutions

- If all components of $\boldsymbol{x}_B$ are strictly positive ($\boldsymbol{x}_B > \mathbf{0}$) then $\boldsymbol{x}$ is a **nondegenerate basic (feasible) solution**, otherwise it is a **degenerate basic (feasible) solution**
- Basic feasible solutions are essential as these give the algebraic interpretation for the geometric notion of extreme points (which we know are key to solving linear programs)
- **Theorem:** $\boldsymbol{x}$ is a basic feasible solution for the linear program $\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \mathbf{0}\}$ if and only if $\boldsymbol{x}$ is an extreme point of the feasible region $X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x} \geq \mathbf{0}\}$
- If, in addition, $\boldsymbol{x}$ is also nondegenerate then it is generated by a single basis
- Iterating along basic feasible solutions (i.e., extreme points) we can solve a linear program to optimality: the simplex method

Basic Feasible Solutions: Example

- Consider the feasible region

$$\begin{array}{rcrcl} x_1 & + & x_2 & \le & 6 \\ & & x_2 & \le & 3 \\ x_1, & & x_2 & \ge & 0 \end{array}$$

- This is in canonical form ("$\le$" type constraints), we need to convert to standard form ("$=$" type constraints)
- Introduce the slack variables x_3 and x_4

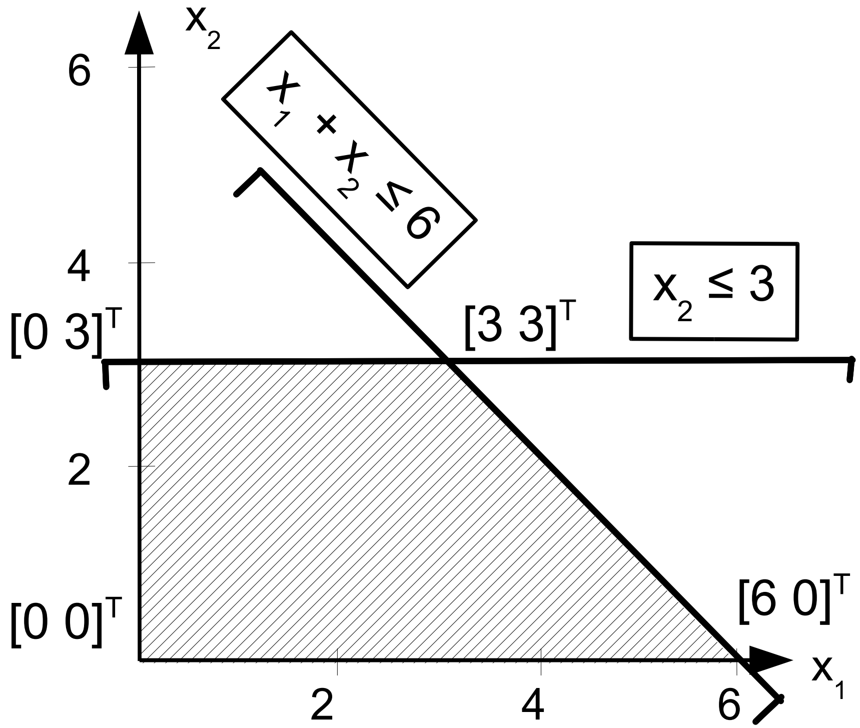
$$\begin{array}{rcrcrcrcl} x_1 & + & x_2 & + & x_3 & & & = & 6 \\ & & x_2 & & & + & x_4 & = & 3 \\ x_1, & & x_2, & & x_3, & & x_4 & \ge & 0 \end{array}$$

Basic Feasible Solutions: Example

- The matrix of the constraint system

$$\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4] = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

- $\boldsymbol{A}$ is of full row rank so the size of the basic matrix $\boldsymbol{B}$ is 2×2 and there can be $\binom{4}{2}$ of them
- Basic feasible solutions are the ones for which $\boldsymbol{B}$ is nonsingular and $\boldsymbol{B}^{-1}\boldsymbol{b} \geq \mathbf{0}$



Basic Feasible Solutions: Example

1. $\boldsymbol{B} = [\boldsymbol{a}_1 \quad \boldsymbol{a}_2] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ gives a basic feasible solution

$$\boldsymbol{x}_{\boldsymbol{B}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$\boldsymbol{x}_{\boldsymbol{N}} = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, extreme point: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

2. $\boldsymbol{B} = [\boldsymbol{a}_1 \quad \boldsymbol{a}_4] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ gives a basic feasible solution

$$\boldsymbol{x}_{\boldsymbol{B}} = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

$\boldsymbol{x}_{\boldsymbol{N}} = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, extreme point: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

Basic Feasible Solutions: Example

3. $\boldsymbol{B} = [\boldsymbol{a}_2 \quad \boldsymbol{a}_3] = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ gives a basic feasible solution

$$\boldsymbol{x_B} = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$\boldsymbol{x_N} = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, extreme point: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$

4. $\boldsymbol{B} = [\boldsymbol{a}_2 \quad \boldsymbol{a}_4] = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ is not a feasible basis!

$$\boldsymbol{x_B} = \begin{bmatrix} x_2 \\ x_4 \end{bmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix} \not\geq \boldsymbol{0}$$

$$\boldsymbol{x_N} = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Basic Feasible Solutions: Example

5. $\boldsymbol{B} = [\boldsymbol{a}_3 \quad \boldsymbol{a}_4] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ gives a basic feasible solution

   $$\boldsymbol{x}_{\boldsymbol{B}} = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

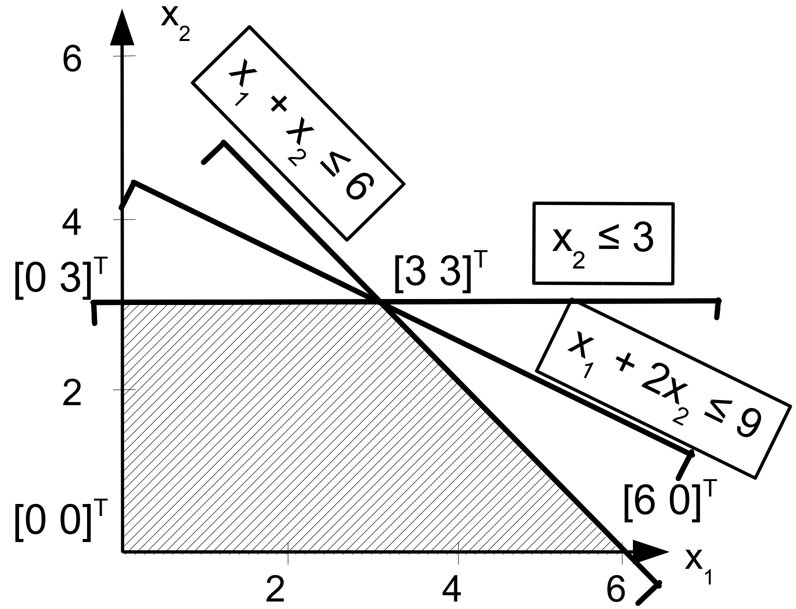
   $\boldsymbol{x}_{\boldsymbol{N}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, extreme point: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

6. $\boldsymbol{B} = [\boldsymbol{a}_1 \quad \boldsymbol{a}_3] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ singular, does not generate a basic solution

Degenerate Basic Solutions: Example

- Introduce a new "redundant" constraint

$$
\begin{array}{ccccc}
x_1 & + & x_2 & \leq & 6 \\
 & & x_2 & \leq & 3 \\
x_1 & + & 2x_2 & \leq & 9 \\
x_1, & & x_2 & \geq & 0
\end{array}
$$



- The constraint system converted to standard form introducing appropriate slack variables x_3, x_4, and x_5

$$
\begin{array}{ccccccccccc}
x_1 & + & x_2 & + & x_3 & & & & & = & 6 \\
 & & x_2 & & & + & x_4 & & & = & 3 \\
x_1 & + & 2x_2 & & & & & + & x_5 & = & 9 \\
x_1, & & x_2, & & x_3, & & x_4 & & x_5 & \geq & 0
\end{array}
$$

Degenerate Basic Solutions: Example

$$\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3, \boldsymbol{a}_4, \boldsymbol{a}_5] = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

- The basic feasible solution generated by the basic matrix $\boldsymbol{B} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_3]$ is degenerate

$$\boldsymbol{x_B} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \not> \boldsymbol{0}$$

$$\boldsymbol{x_N} = \begin{bmatrix} x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$, extreme point: $$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Degenerate Basic Solutions: Example

- Similarly, $\boldsymbol{B} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \boldsymbol{a}_4]$ also generates a degenerate basic feasible solution

$$\boldsymbol{x}_{\boldsymbol{B}} = \begin{bmatrix} x_1 \\ x_2 \\ x_4 \end{bmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 0 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} \not> \boldsymbol{0}$$

$$\boldsymbol{x}_{\boldsymbol{N}} = \begin{bmatrix} x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \text{ extreme point: } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

- Two basic feasible solutions give the same extreme point
- In 2 dimensions extreme points are generated by 2 hyperplanes
- At the extreme point $\boldsymbol{x} = \left[\begin{smallmatrix} 3 \\ 3 \end{smallmatrix}\right]$ three hyperplanes meet!

The Simplex Method

- As shown, if a linear program is solvable then at least one optimal solution is guaranteed to occur at an extreme point of the feasible region
- Extreme points correspond to basic feasible solutions
- Unfortunately there can be $\binom{n}{m}$ of these, cannot generate all not even in moderate dimensions
- The simplex method, starting from an initial basic feasible solution, generates new basic feasible solutions iteratively that improve the objective function value
- In practice the simplex visits only a modest number of extreme points to find the optimal solution or prove unboundedness

The Simplex Method

- Let $\boldsymbol{A}$ be an $m \times n$ matrix with $\mathrm{rank}(\boldsymbol{A}) = \mathrm{rank}(\boldsymbol{A}, \boldsymbol{b}) = m$, $\boldsymbol{b}$ be a column m-vector, $\boldsymbol{x}$ be a column n-vector, and $\boldsymbol{c}^T$ be a row n-vector, and consider the linear program

$$\begin{aligned} z = \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- Let $\boldsymbol{B}$ be an initial basis and $\boldsymbol{x} = \begin{bmatrix} \boldsymbol{x}_B \\ \boldsymbol{x}_N \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}^{-1}\boldsymbol{b} \\ \boldsymbol{0} \end{bmatrix} \geq \boldsymbol{0}$ be the **basic feasible solution** for $\boldsymbol{B}$ and write $\boldsymbol{A} = [\boldsymbol{B} \quad \boldsymbol{N}]$
- Write the linear program in the space of nonbasic variables
- This way we obtain the explicit expression of the the basic variables and the objective function value in terms of the nonbasic variables

The Simplex Method

- The constraint system in the basis $\boldsymbol{B}$:

$$\boldsymbol{A}\boldsymbol{x} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{N} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_{\boldsymbol{B}} \\ \boldsymbol{x}_{\boldsymbol{N}} \end{bmatrix} = \boldsymbol{b}, \text{ that is } \boldsymbol{B}\boldsymbol{x}_{\boldsymbol{B}} + \boldsymbol{N}\boldsymbol{x}_{\boldsymbol{N}} = \boldsymbol{b}$$

$$\boldsymbol{x}_{\boldsymbol{B}} = \boldsymbol{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x}_{\boldsymbol{N}} \qquad (*)$$

- Let N denote the set of nonbasic variables and denote the columns of the matrix $\boldsymbol{B}^{-1}\boldsymbol{N}$ by $\boldsymbol{y}_j$ for each $j \in N$, and let $\bar{\boldsymbol{b}} = \boldsymbol{B}^{-1}\boldsymbol{b}$
- The basic variables in the nonbasic variable space:

$$\boldsymbol{x}_{\boldsymbol{B}} = \bar{\boldsymbol{b}} - \sum_{j \in N} \boldsymbol{y}_j x_j$$

The Simplex Method

- Reorder the objective function so that the first m coefficients belong to the basic variables and the remaining $n-m$ variables to the nonbasic variables: $\boldsymbol{c}^T = [\boldsymbol{c_B}^T \quad \boldsymbol{c_N}^T]$
- Substituting (*):

$$z = \boldsymbol{c}^T\boldsymbol{x} = [\boldsymbol{c_B}^T \quad \boldsymbol{c_N}^T]\begin{bmatrix}\boldsymbol{x_B}\\ \boldsymbol{x_N}\end{bmatrix} = \boldsymbol{c_B}^T\boldsymbol{x_B} + \boldsymbol{c_N}^T\boldsymbol{x_N} =$$

$$\boldsymbol{c_B}^T(\mathrm{B}^{-1}\boldsymbol{b} - \boldsymbol{B}^{-1}\boldsymbol{N}\boldsymbol{x_N}) + \boldsymbol{c_N}^T\boldsymbol{x_N} =$$

$$\boldsymbol{c_B}^T\mathrm{B}^{-1}\boldsymbol{b} + (\boldsymbol{c_N}^T - \boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{N})\boldsymbol{x_N}$$

- Let $z_0 = \boldsymbol{c_B}^T\mathrm{B}^{-1}\boldsymbol{b}$ and denote the components of $\boldsymbol{c_N}^T - \boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{N}$ by z_j for each $j \in N$

$$z = z_0 + \sum_{j\in N} z_j x_j$$

The Simplex Method

- **Theorem:** the **linear program in the nonbasic variable space** is given by

$$\begin{aligned} \max \quad & z_0 + \sum_{j \in N} z_j x_j \\ \text{s.t.} \quad & \boldsymbol{x_B} = \bar{\boldsymbol{b}} - \sum_{j \in N} \boldsymbol{y}_j x_j \\ & \boldsymbol{x_B}, \boldsymbol{x_N} \geq \mathbf{0} \end{aligned}$$

  where
  - N denotes the set of nonbasic variables
  - $\bar{\boldsymbol{b}} = \boldsymbol{B}^{-1}\boldsymbol{b}$
  - $\boldsymbol{y}_j$ denotes the column of the matrix $\boldsymbol{B}^{-1}\boldsymbol{N}$ that belongs to the j-th nonbasic variable
  - $z_0 = \boldsymbol{c_B}^T \mathrm{B}^{-1}\boldsymbol{b} = \boldsymbol{c_B}^T\bar{\boldsymbol{b}}$
  - z_j is the component of the row vector $\boldsymbol{c_N}^T - \boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{N}$ that belongs to the j-th nonbasic variable

The Simplex Method: Example

- Consider the linear program

$$\begin{array}{lcccccc} \max & x_1 & + & x_2 & & \\ \text{s.t.} & x_1 & + & 2x_2 & \leq & 4 \\ & & & x_2 & \leq & 1 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

- Introducing slack variables and converting to standard form:

$$\begin{array}{rcl} \max \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix} \boldsymbol{x} & & \\ \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \boldsymbol{x} & = & \begin{bmatrix} 4 \\ 1 \end{bmatrix} \\ x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

The Simplex Method: Example

- Let $\boldsymbol{B} = [\boldsymbol{a}_2, \boldsymbol{a}_3]$, then $B = \{2, 3\}$, $N = \{1, 4\}$

  $\boldsymbol{B} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$, $\boldsymbol{B}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$, $\boldsymbol{N} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

  $\boldsymbol{c_B}^T = [1 \quad 0]$, $\boldsymbol{c_N}^T = [1 \quad 0]$

  $$\bar{\boldsymbol{b}} = \mathrm{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

  $$\boldsymbol{B}^{-1}\boldsymbol{N} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

  $$z_0 = \boldsymbol{c_B}^T\bar{\boldsymbol{b}} = [1 \quad 0] \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 1$$

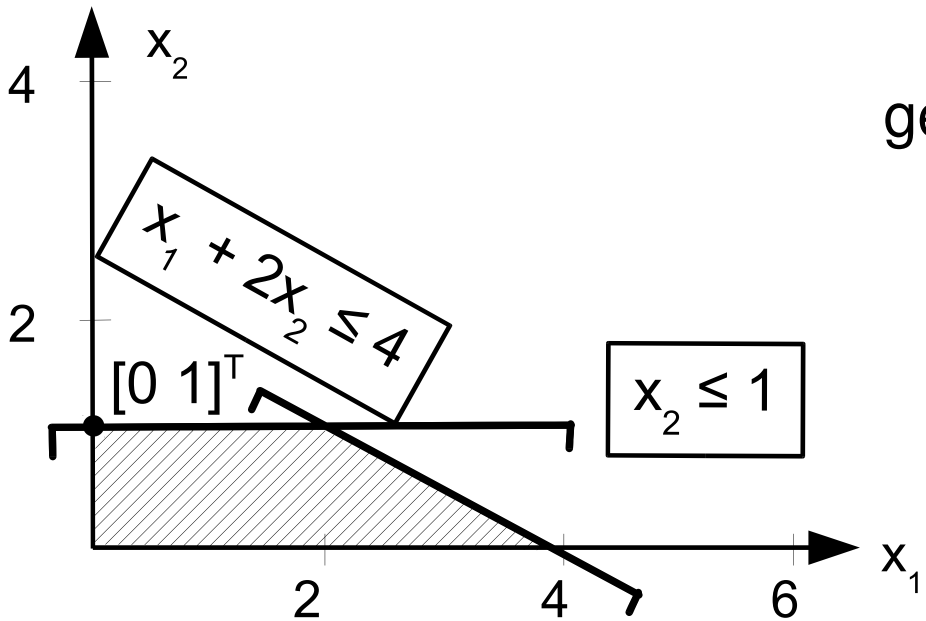
  $$\boldsymbol{c_N}^T - \boldsymbol{c_B}^T\boldsymbol{B}^{-1}\boldsymbol{N} = [1 \quad 0] - [1 \quad 0] \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

  $$[1 \quad 0] - [0 \quad 1] = [1 \quad -1]$$

The Simplex Method: Example

- The linear program in the nonbasic variable space:

$$
\begin{aligned}
\max \quad & 1 + x_1 - x_4 \\
\text{s.t.} \quad & \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_1 - \begin{bmatrix} 1 \\ -2 \end{bmatrix} x_4 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$



The basic feasible solution generated by the current basis matrix $\boldsymbol{B}$:

$$
\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}
$$

The Simplex Method: Pivot

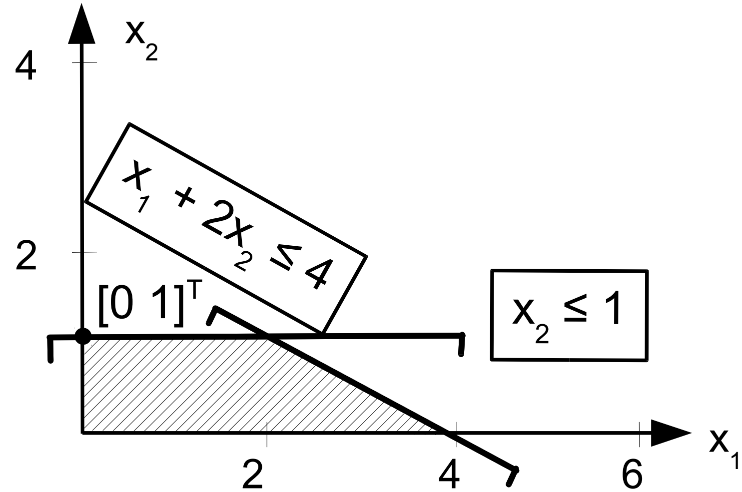
- The linear program in the nonbasic variable space:

$$\begin{aligned} \max \quad & z_0 + \sum_{j \in N} z_j x_j \\ \text{s.t.} \quad & \boldsymbol{x_B} = \bar{\boldsymbol{b}} - \sum_{j \in N} \boldsymbol{y}_j x_j \\ & \boldsymbol{x_B}, \boldsymbol{x_N} \geq \mathbf{0} \end{aligned}$$

- Since $\begin{bmatrix} \boldsymbol{x_B} \\ \boldsymbol{x_N} \end{bmatrix}$ is a basic solution we have $\boldsymbol{x_N} = \mathbf{0}$
- The importance of the above form is that it specifies
  - the value of the objective function z_0 and the basic variables $\boldsymbol{x}_B = \bar{\boldsymbol{b}}$ in the current basis, and
  - the way basic variables and the objective function would change if we started to increase some nonbasic variable from zero

The Pivot: Example

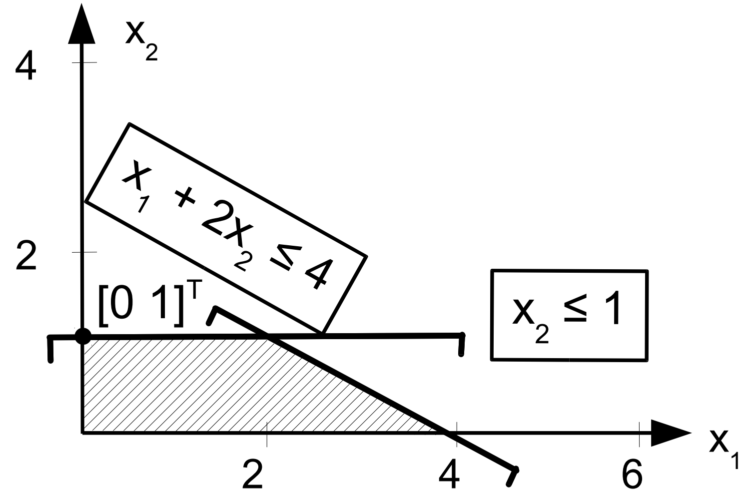
$$\begin{array}{ll} \max & 1 + x_1 - x_4 \\ \text{s.t.} & \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_1 - \begin{bmatrix} 1 \\ -2 \end{bmatrix} x_4 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$



- For instance, if we increased x_4 from zero to 1 while leaving x_1 unchanged at 0
  - the objective function value would decrease from 1 to zero as $z_4 = -1$
  - x_2 would decrease from 1 to zero since $y_{24} = 1$
  - x_3 would increase from 2 to 4 as $y_{34} = -2$
- It is not worth increasing x_4 as this would reduce the objective function value!

The Pivot: Example

$$\begin{array}{ll} \max & 1 + x_1 - x_4 \\ \text{s.t.} & \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_1 - \begin{bmatrix} 1 \\ -2 \end{bmatrix} x_4 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$



- If we increased other nonbasic variable, x_1, from zero to 1
  - the objective function would grow from 1 to 2 by $z_1 = 1$
  - x_2 would not change since $y_{21} = 0$
  - x_3 would fall from 2 to 1 as $y_{31} = 1$
- Worth increasing x_1 as this increases the objective function!
- Increase x_1 until some basic variable becomes negative
- For $x_1 = 2$, x_3 changes to 0 and negative for any $x_1 > 2$

The Simplex Method: Pivot

- Consider the nonbasic variable x_k whose expansion would produce the largest gain in the objective function value

$$k = \underset{j \in N}{\operatorname{argmax}}\, z_j$$

- Fix every other nonbasic variable at zero and increase x_k

$$\begin{aligned} z &= z_0 + z_k x_k \\ \begin{bmatrix} x_{B_1} \\ x_{B_2} \\ \vdots \\ x_{B_m} \end{bmatrix} &= \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \vdots \\ \bar{b}_m \end{bmatrix} - \begin{bmatrix} y_{1k} \\ y_{2k} \\ \vdots \\ y_{mk} \end{bmatrix} x_k \end{aligned}$$

The Simplex Method: Pivot

- Nonbasic variable x_k can be increased until some basic variable drops to zero
- Let $x_i : i \in B$ some basic variable for which $y_{ik} > 0$
- Increasing the nonbasic variable x_k, x_i is nonnegative as long as

$$0 \leq x_i = \bar{b}_i - y_{ik} x_k$$
$$x_k \leq \frac{\bar{b}_i}{y_{ik}}$$

- The first basic variable x_r that drops to zero:

$$r = \operatorname*{argmin}_{i \in \{1,\dots,m\}} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$$

The Simplex Method: Pivot

- Let the current basis be nongenenerate ($\bar{\boldsymbol{b}} > \mathbf{0}$) and assume that $\exists k \in N : z_k > 0$ and $\exists i \in B : y_{ik} > 0$
- Let $k = \underset{j \in N}{\operatorname{argmax}}\, z_j$ and $r = \underset{i \in B}{\operatorname{argmin}} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$
- Increasing x_k from zero to $\frac{\bar{b}_r}{y_{rk}}$:

$$
\begin{aligned}
z &= z_0 + z_k \frac{\bar{b}_r}{y_{rk}} & \\
x_k &= \frac{\bar{b}_r}{y_{rk}} > 0, x_r = 0 & \\
x_{B_i} &= \bar{b}_i - \frac{y_{ik}}{y_{rk}} \bar{b}_r & i \in B \setminus \{r\} \\
x_j &= 0 & j \in N \setminus \{k\}
\end{aligned}
$$

- The objective function value grows, i.e., $z > z_0$ as $z_k \frac{\bar{b}_r}{y_{rk}} > 0$

The Simplex Method: Pivot

- This transformation is called the **pivot**, during which
  - the nonbasic variable x_k increases from zero and **enters** the basis
  - the basic variable x_r drops to zero and **leaves** the basis
  - all remaining nonbasic variables remain zero and all remaining basic variables remain nonnegative
- **Theorem:** the pivot results a new basic feasible solution
- The proof (omitted here) would go by observing that it is enough to show that the new basis is nonsingular, since all variables remain nonnegative duting the pivot
- Then using a basic result from linear algebra to show that this is guaranteed by the choice for r:

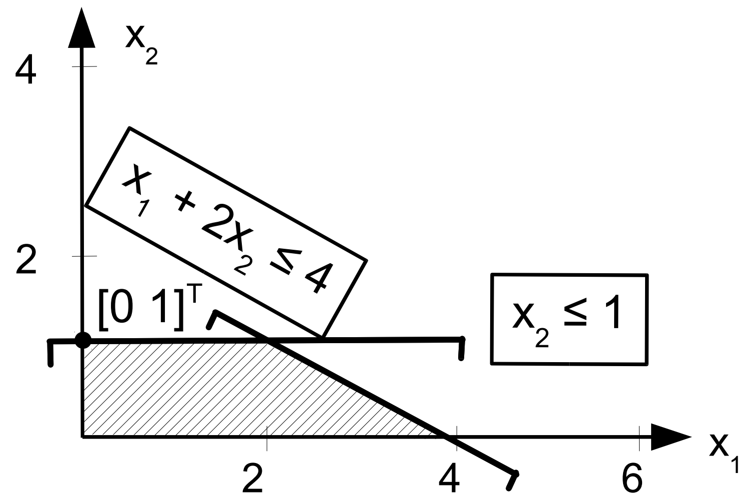
$$r = \operatorname*{argmin}_{i \in \{1,\dots,m\}} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$$

The Pivot: Example

$$\max \quad 1 + x_1 - x_4$$

$$\text{s.t.} \quad \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_1 - \begin{bmatrix} 1 \\ -2 \end{bmatrix} x_4$$

$$x_1, x_2, x_3, x_4 \geq 0$$



- x_1 enters the basis since

$$z_1 = \max_{j \in N} z_j = \max\{1, -1\} = 1$$

- x_3 leaves the basis as

$$\frac{\bar{b}_3}{y_{31}} = \min_{i \in B} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\} = \min\{2\} = 2$$

The Pivot: Example

The new point obtained after the pivot:

$k = 1, r = 3$

$B = \{1, 2\}, N = \{3, 4\}$

$z = z_0 + z_k \frac{\bar{b}_r}{y_{rk}} = 1 + 2 = 3$

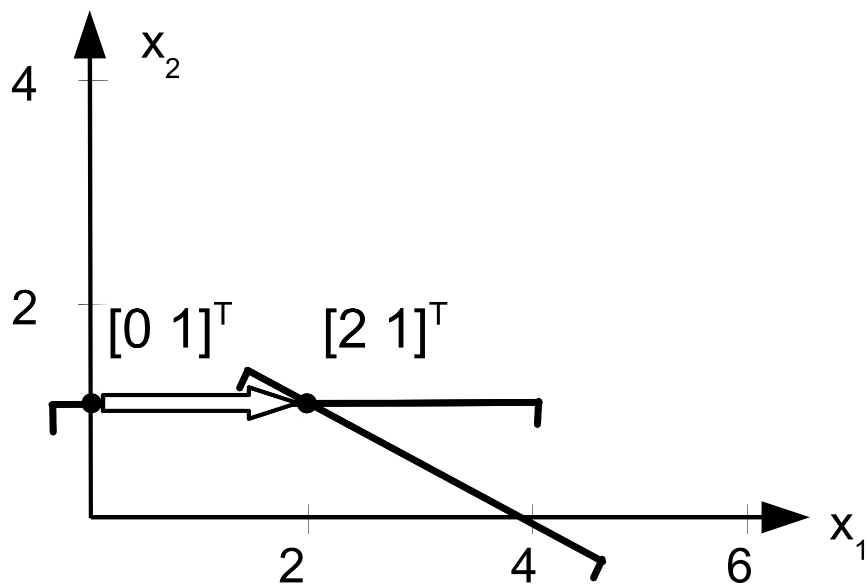
$x_1 = \frac{\bar{b}_r}{y_{rk}} = 2$

$x_2 = \bar{b}_2 - \frac{y_{2k}}{y_{3k}} \bar{b}_3 = 1 - 0 = 1$

$x_3 = \bar{b}_3 - \frac{y_{3k}}{y_{3k}} \bar{b}_3 = 2 - 2 = 0$

$x_4 = 0$

$\boldsymbol{x} = [2 \quad 1 \quad 0 \quad 0]^T$



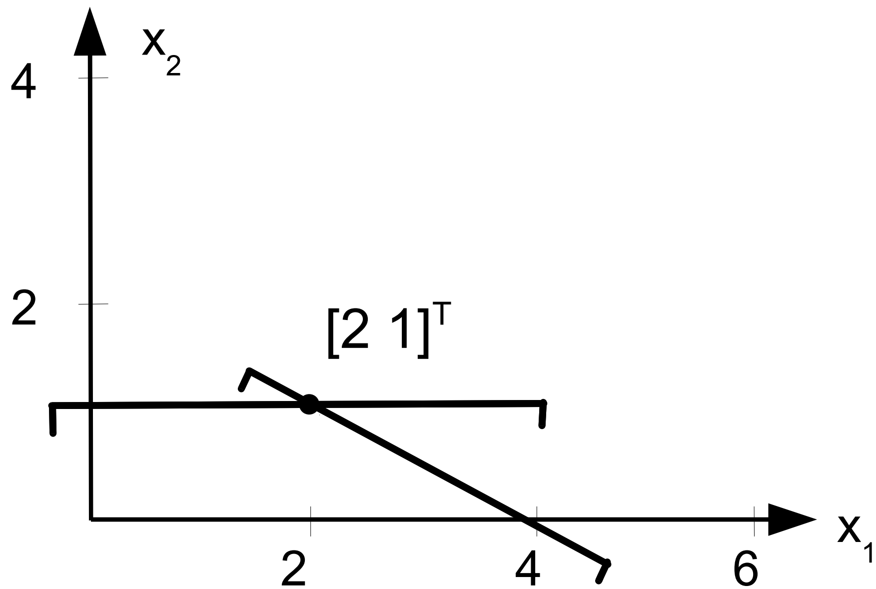
The Pivot: Example

$$B = \{1, 2\}, N = \{3, 4\},$$

$$\boldsymbol{B} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \boldsymbol{N} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$\boldsymbol{B}^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{c_B}^T = [1 \quad 1], \boldsymbol{c_N}^T = [0 \quad 0]$$



- The linear program in the new basis:

$$\begin{aligned} \max \quad & 3 - x_3 + x_4 \\ \text{s.t.} \quad & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_3 - \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_4 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

The Simplex Method: Optimality

- Recall: the pivot rules
  - x_k can enter the basis if $z_k > 0$
  - x_r leaves the basis if $r = \underset{i \in \{1,\ldots,m\}}{\operatorname{argmin}} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$
- If for all $j \in N : z_j \leq 0$ holds then it is not worth increasing any of the nonbasic variables, since the objective function value could not be increased this way
- **Theorem:** the optimality condition for the simplex method

$$\forall j \in N : z_j \leq 0$$

- **Proof:** if for some basic feasible solution $\boldsymbol{x}$ we have $\forall j \in N : z_j \leq 0$, then $\boldsymbol{x}$ is a local optimum
- Since the feasible region is convex and the objective function is concave, $\boldsymbol{x}$ is also a global optimum $\square$

Termination with Optimality: Example

- Continuing with our running example, in the current basis:

$$\begin{array}{rc} \max & 3 - x_3 + x_4 \\ \text{s.t.} & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_3 - \begin{bmatrix} -2 \\ 1 \end{bmatrix} x_4 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

- This basic feasible solution is not optimal since $z_4 > 0$
- Correspondingly, in the next pivot
  - x_4 enters the basis, and
  - x_2 leaves the basis because

$$\frac{\bar{b}_2}{y_{24}} = \min_{i \in B} \left\{ \frac{\bar{b}_i}{y_{i4}} : y_{i4} > 0 \right\} = \min\{1\} = 1$$

Termination with Optimality: Example

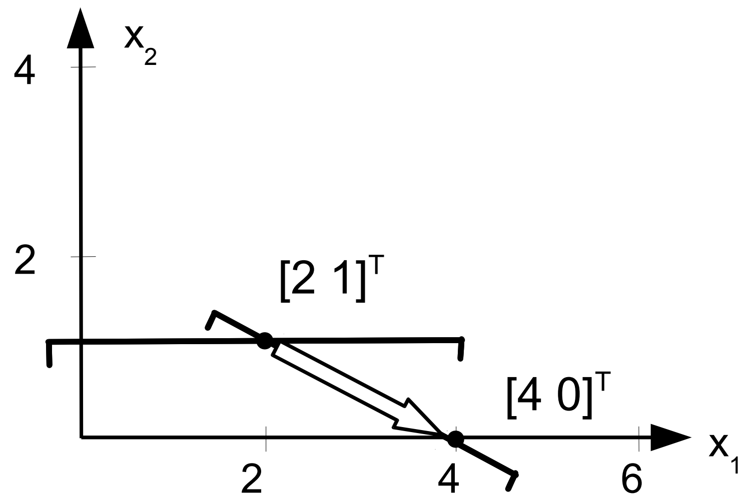
- The linear program in the new basis:

$$B = \{1,4\}, N = \{2,3\}, \boldsymbol{c_B}^T = [1 \quad 0], \boldsymbol{c_N}^T = [1 \quad 0]$$

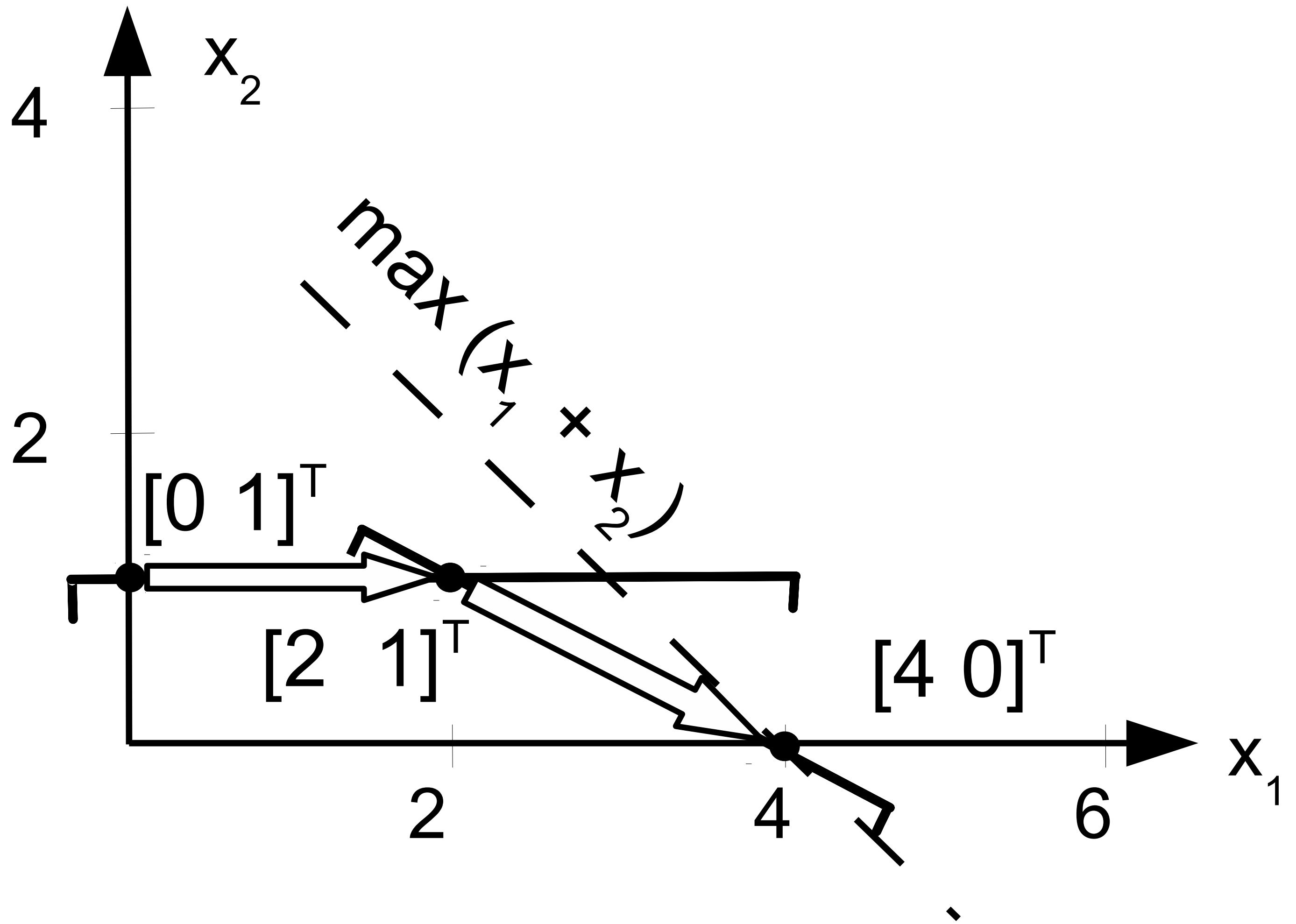
$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \boldsymbol{N} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \boldsymbol{B}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- The basic feasible solution $\boldsymbol{x} = [4 \quad 0 \quad 0 \quad 1]^T$ is optimal

$$\begin{aligned} \max \quad & 4 - x_2 - x_3 \\ \text{s.t.} \quad & \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_2 - \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$



The Simplex Method: Geometry



Termination in a Unique Optimum

- **Definition:** the feasible solution $\bar{\boldsymbol{x}}$ to the linear program $\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ is a **unique optimal solution** if for each $\boldsymbol{x} \neq \bar{\boldsymbol{x}}$ feasible solution $\boldsymbol{c}^T\boldsymbol{x} < \boldsymbol{c}^T\bar{\boldsymbol{x}}$
- **Theorem:** the basic feasible solution $\bar{\boldsymbol{x}}$ is a unique optimal solution if $\forall j \in N : z_j < 0$
- **Proof:** let $\boldsymbol{x}$ be any feasible solution different from $\bar{\boldsymbol{x}}$ and let the corresponding objective function value be z
- Let N denote the set of nonbasic variables corresponding to $\bar{\boldsymbol{x}}$, then

$$z = z_0 + \sum_{j \in N} z_j x_j$$

- We observe that there is $j \in N$ so that $x_j > 0$ (otherwise $\boldsymbol{x} = \bar{\boldsymbol{x}}$) and from $\forall j \in N : z_j < 0$ it follows that $z < z_0$ $\square$

Unique Optimum: Example

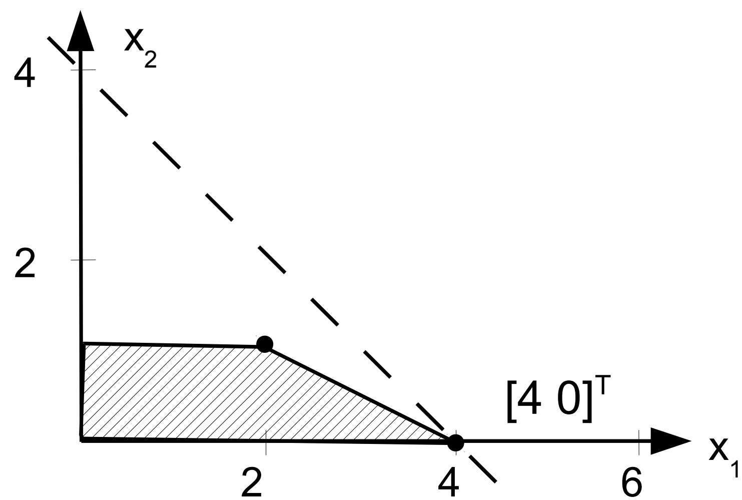
- Consider the basic feasible solution $\boldsymbol{x} = [4 \quad 0 \quad 0 \quad 1]^T$ for the running example

$$B = \{1, 4\}, N = \{2, 3\}$$

$$\boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \boldsymbol{N} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}, \boldsymbol{B}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\boldsymbol{c_B}^T = [1 \quad 0], \boldsymbol{c_N}^T = [1 \quad 0]$$

$$\begin{aligned} \max \quad & 4 - x_2 - x_3 \\ \text{s.t.} \quad & \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \end{bmatrix} x_2 - \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$



- $z_2 < 0$, $z_3 < 0$: $\boldsymbol{x}$ is unique optimum

Termination with Alternative Optima

- Suppose that $\bar{\boldsymbol{x}}$ is an optimal basic feasible solution (i.e., $\forall j \in N : z_j \leq 0$), but there is a nonbasic variable, say, k, for which the optimality condition holds with equality: $z_k = 0$
- If $\bar{\boldsymbol{x}}$ is nondegenerate then x_k can be increased from zero by some $\epsilon > 0$ small enugh so that no basic variable becomes negative

$$
\begin{aligned}
z &= z_0 + z_k x_k = z_0 + 0 x_k \\
\begin{bmatrix} x_{B_1} \\ \vdots \\ x_{B_m} \end{bmatrix} &= \begin{bmatrix} \bar{b}_1 \\ \vdots \\ \bar{b}_m \end{bmatrix} - \begin{bmatrix} y_{1k} \\ \vdots \\ y_{mk} \end{bmatrix} x_k
\end{aligned}
$$

- Any choice $0 < x_k \leq \epsilon$ yields alternative optimal solutions since the objective function value does not change due to $z_k = 0$

Termination with Alternative Optima

- The linear program as the function of the nonbasic variable x_k, let $z_k = 0$ and all other $z_j \leq 0$

$$\begin{aligned} \max \quad & z_0 + 0x_k \\ \text{s.t.} \quad & \boldsymbol{x} = \begin{bmatrix} \boldsymbol{x_B} \\ \boldsymbol{x_N} \end{bmatrix} = \begin{bmatrix} \bar{\boldsymbol{b}} \\ \boldsymbol{0} \end{bmatrix} + \begin{bmatrix} -\boldsymbol{y}_k \\ \boldsymbol{e}_k \end{bmatrix} x_k \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

  where, as usual, $\boldsymbol{y}_k$ is the column of $\boldsymbol{B}^{-1}\boldsymbol{N}$ that belongs to k and $\boldsymbol{e}_k$ is the k-th canonical unit vector
- We obtain alternative optimal as long as $\boldsymbol{x_B} \geq \boldsymbol{0}$

$$\boldsymbol{x} = \begin{bmatrix} \bar{\boldsymbol{b}} \\ \boldsymbol{0} \end{bmatrix} + \lambda \begin{bmatrix} -\boldsymbol{y}_k \\ \boldsymbol{e}_k \end{bmatrix} \qquad 0 \leq \lambda \leq \min_{i \in B} \left\{ \frac{\bar{b}_i}{y_{ik}} : y_{ik} > 0 \right\}$$

Alternative Optima: Example

- Consider the linear program

$$\begin{array}{lccccc} \max & 2x_1 & + & 4x_2 & & \\ \text{s.t.} & x_1 & + & 2x_2 & \leq & 4 \\ & -x_1 & + & x_2 & \leq & 1 \\ & x_1, & & x_2, & \geq & 0 \end{array}$$

- Introducing slack variables to convert to standard form:

$$\begin{array}{lccccccccc} \max & 2x_1 & + & 4x_2 & & & & & & \\ \text{s.t.} & x_1 & + & 2x_2 & + & x_3 & & & = & 4 \\ & -x_1 & + & x_2 & & & + & x_4 & = & 1 \\ & x_1, & & x_2, & & x_3, & & x_4 & \geq & 0 \end{array}$$

Alternative Optima: Example

- Choose the basis as $\boldsymbol{B} = [\boldsymbol{a}_1 \quad \boldsymbol{a}_4] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$,

  $$\boldsymbol{B}^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

- The corresponding basic feasible solution

  $$\boldsymbol{x_B} = \bar{\boldsymbol{b}} = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \boldsymbol{B}^{-1}\boldsymbol{b} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

  $$\boldsymbol{x_N} = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Alternative Optima: Example

- The parameters corresponding to this basis

$$\boldsymbol{c_B}^T = [2 \quad 0], \boldsymbol{c_N}^T = [4 \quad 0]$$

$$\boldsymbol{B}^{-1}\boldsymbol{N} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$$

$$z_0 = \boldsymbol{c_B}^T \bar{\boldsymbol{b}} = [2 \quad 0] \begin{bmatrix} 4 \\ 5 \end{bmatrix} = 8$$

$$\boldsymbol{c_N}^T - \boldsymbol{c_B}^T \boldsymbol{B}^{-1}\boldsymbol{N} = [4 \quad 0] - [2 \quad 0] \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$$

$$= [4 \quad 0] - [4 \quad 2] = [0 \quad -2]$$

Alternative Optima: Example

- The linear program in the nonbasic variable space:

$$\begin{array}{ll} \max & 8 + 0x_2 - 2x_3 \\ \text{s.t.} & \begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \end{bmatrix} x_2 - \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_3 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

- Alternative optimal solutions:

$$\boldsymbol{x} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 5 \end{bmatrix} + \lambda \begin{bmatrix} -2 \\ 1 \\ 0 \\ -3 \end{bmatrix} : 0 < \lambda \leq \frac{5}{3}$$

- All convex combinations of the points $[4 \quad 0]^T$ and $[\frac{2}{3} \quad \frac{5}{3}]^T$