Solving Linear Programs: The Basics
A Summary

WARNING: this is just a summary of the material covered in the full slide-deck ***Solving Linear Programs: The Basics*** *that will orient you as per the topics covered there; you are required to learn the full version, not just this summary!*

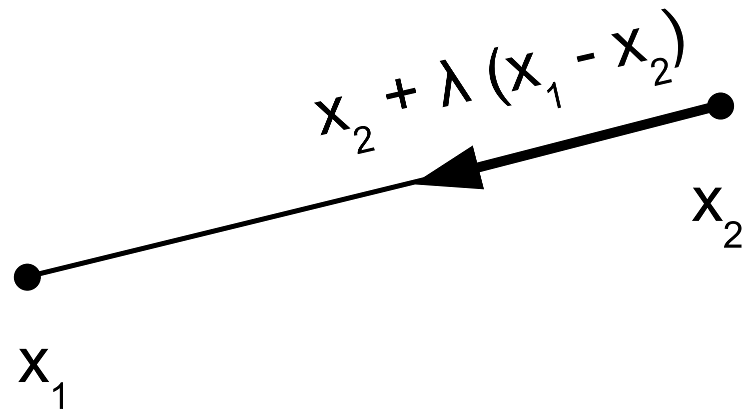
- Introduction to convex analysis
- Convex and concave functions, the fundamental theorem of convex programming
- Convex geometry: polyhedra, the Minkowski-Weyl theorem (the Representation Theorem)
- Solving linear programs using the Minkowski-Weyl theorem
- Solving simple linear programs with the graphical method
- The feasible region (bounded, unbounded, empty) and optimal solutions (unique, alternative, unbounded)

Convex Sets

- For each $0 \leq \lambda \leq 1$, the points arising as

$$\lambda \boldsymbol{x}_1 + (1 - \lambda)\boldsymbol{x}_2$$

  are called the **convex combinations of vectors $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$**
- Geometrically, the convex combinations of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ span the **line segment** between point $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$

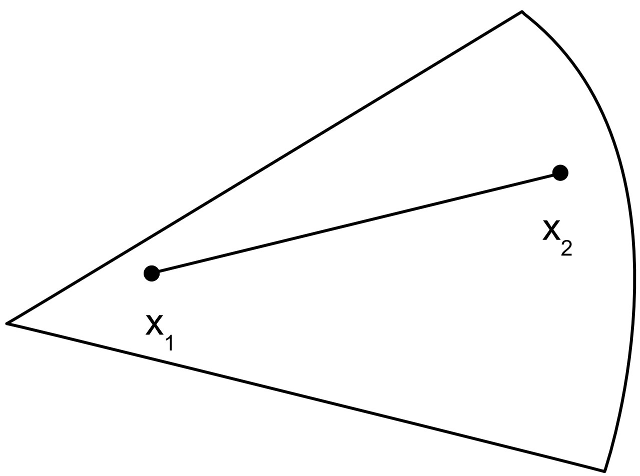


Convex Sets

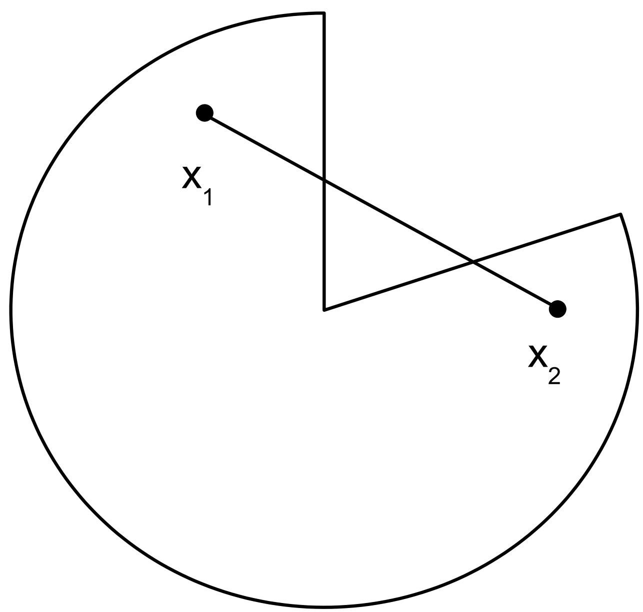
- A set $X \subset \mathbb{R}^n$ is **convex** if for each points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in X it holds that

$$\forall \lambda \in [0, 1] : \lambda \boldsymbol{x}_1 + (1 - \lambda)\boldsymbol{x}_2 \in X$$

- In other words, X is convex if it contains all convex combinations of each of its points



Convex set



Nonconvex set

Convex Sets: Examples

- The **convex combinations of** k **points** $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k$:

$$X = \left\{ \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i : \sum_{i=1}^{k} \lambda_i = 1, \forall i \in \{1, \ldots, k\} : \lambda_i \geq 0 \right\}$$

$$X = \operatorname{conv}\{\boldsymbol{x}_i : 1 \leq i \leq k\}$$

- The 3-sphere: $X = \{[x, y, z] : x^2 + y^2 + x^2 \leq 1\}$
- Vector space: $X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\}$
- Affine space (translated vector space): $X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$
- **Feasible region of a linear program:**

$$X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$$

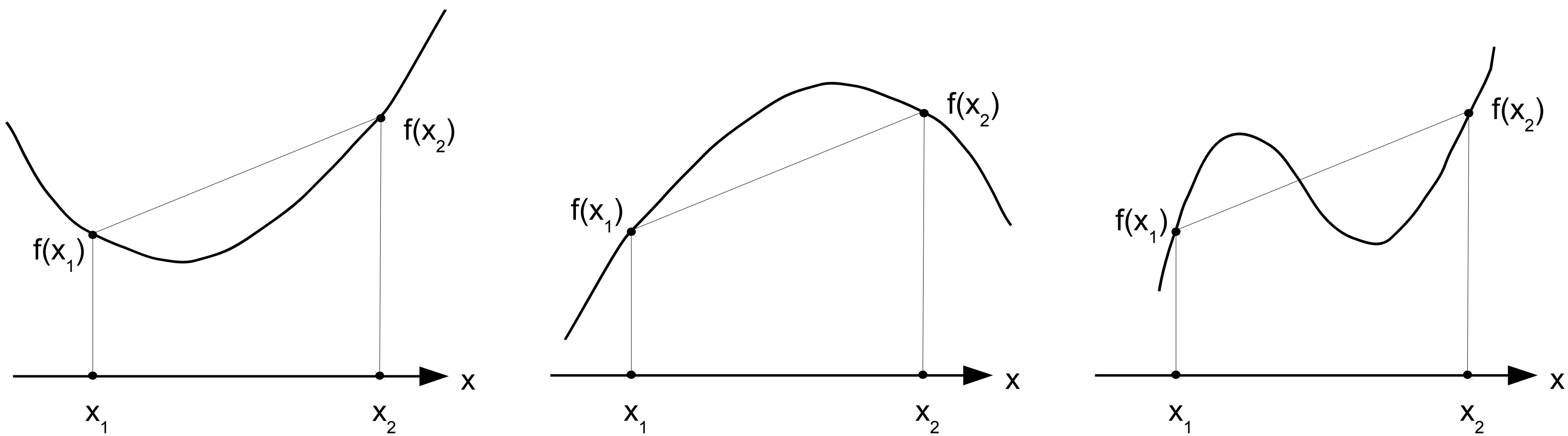
$$X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$$

Convex and Concave Functions

- A function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is **convex** on a convex set $X \subseteq \mathbb{R}^n$ if for each $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in X:

$$f(\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) \leq \lambda f(\boldsymbol{x}_1) + (1-\lambda) f(\boldsymbol{x}_2) \quad \forall \lambda \in [0,1]$$

- The line segment between any two points $f(\boldsymbol{x}_1)$ and $f(\boldsymbol{x}_2)$ on the graph of the function lies above or on the graph
- Function f is **concave** if $(-f)$ is convex

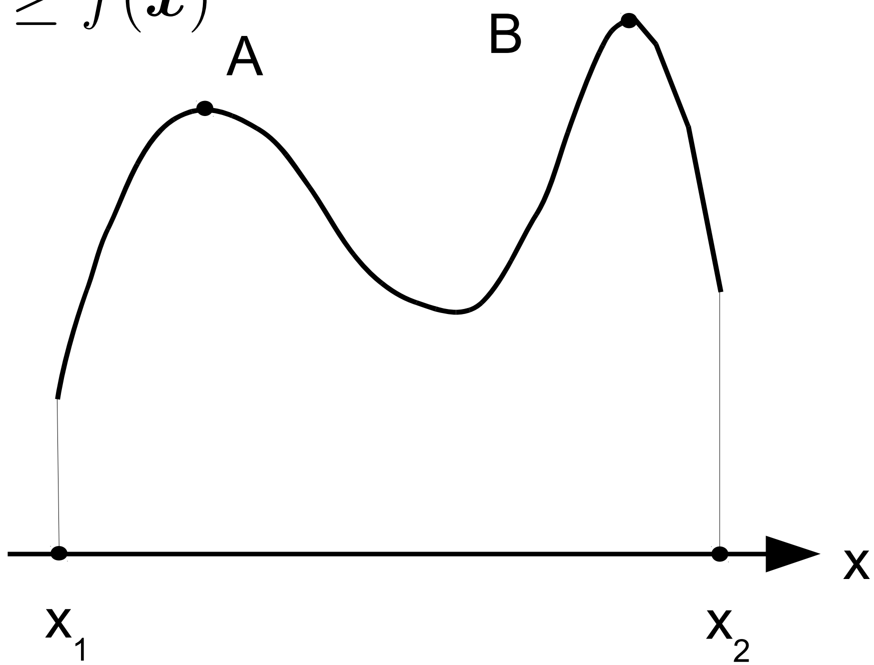


(a) convex (b) concave (c) neither

Optimization on a Convex Set

- Given function $f : \mathbb{R}^n \mapsto \mathbb{R}$ and set X, solve the generic optimization problem $\max f(\boldsymbol{x}) : \boldsymbol{x} \in X$
- Some $\bar{\boldsymbol{x}} \in X$ is a **global optimal solution** (or global optimum) if for each $\boldsymbol{x} \in X$: $f(\bar{\boldsymbol{x}}) \geq f(\boldsymbol{x})$
- An $\bar{\boldsymbol{x}} \in X$ is a **local optimum** if there is a neighborhood $N_\epsilon(\bar{\boldsymbol{x}})$ of $\bar{\boldsymbol{x}}$ (an open ball of radius $\epsilon > 0$ with centre $\bar{\boldsymbol{x}}$) so that $\forall \boldsymbol{x} \in N_\epsilon(\bar{\boldsymbol{x}}) \cap X$: $f(\bar{\boldsymbol{x}}) \geq f(\boldsymbol{x})$

point A is a local optimum and point B is a global optimum on the closed interval $[x_1, x_2]$

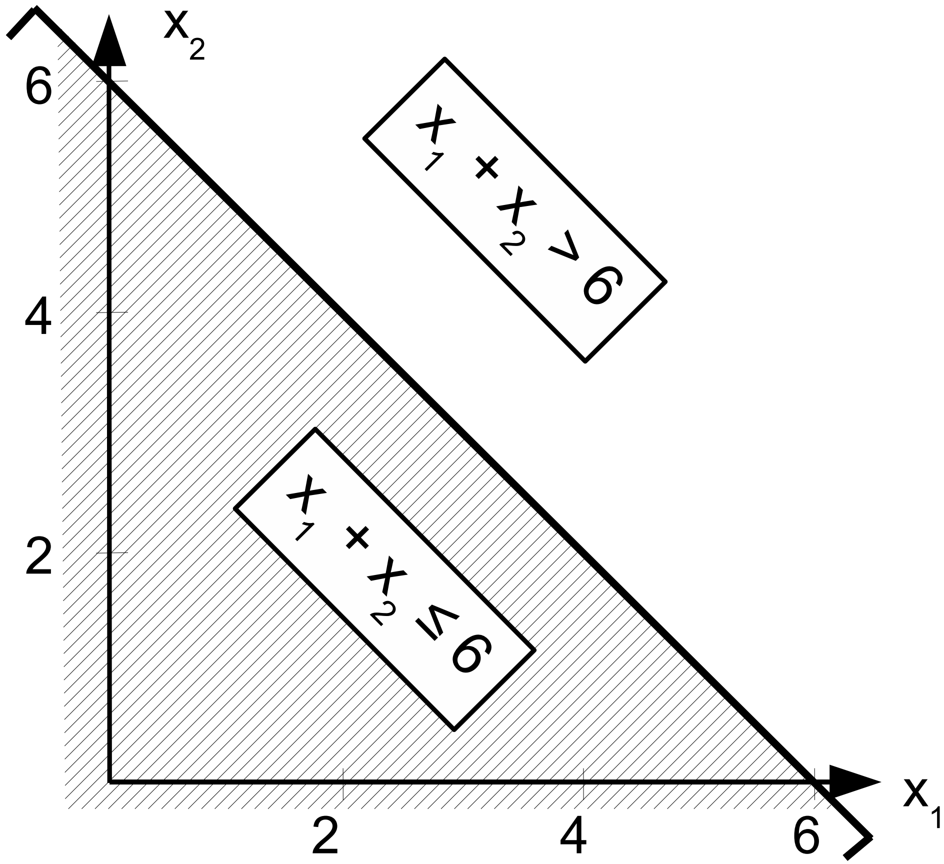


Optimization on a Convex Set

- **Fundamental Theorem of Convex Programming:** Let X be a nonempty **convex set** in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a **concave function** on X. Consider the optimization problem $\max f(\boldsymbol{x}) : \boldsymbol{x} \in X$. Then, if $\bar{\boldsymbol{x}} \in X$ is a local optimal solution then it is also a global optimum
- **Proof:** in the slide-deck, please understand and learn!
- **Bottomline:** the Fundamental Theorem sets apart “simple” (provably polynomial-time solvable) from “complex” (hopeless, intractable) problems
- **Convex program:** minimization of a convex objective function over a convex set $=$ maximization of a concave objective function over a convex set

Hyperplanes and Half-spaces

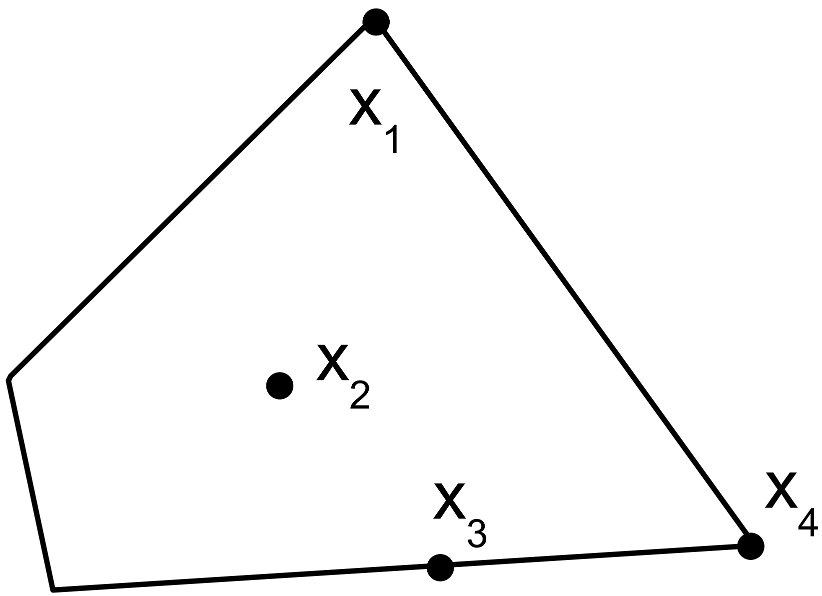
- **Hyperplane:** all $\boldsymbol{x} \in \mathbb{R}^n$ satisfying the equation $\boldsymbol{a}^T\boldsymbol{x} = b$ for some $\boldsymbol{a}^T$ row n-vector (the **normal vector**) and scalar b
- The hyperplane $X = \{\boldsymbol{x} : \boldsymbol{a}^T\boldsymbol{x} = b\}$ divides the space $\mathbb{R}^n$ into two **half-spaces**
  - "lower" half-space: $\{\boldsymbol{x} : \boldsymbol{a}^T\boldsymbol{x} \leq b\}$
  - "upper" half-space: $\{\boldsymbol{x} : \boldsymbol{a}^T\boldsymbol{x} > b\}$
- Hyperplanes and half-spaces are convex



Extreme Points

- Given a convex set X, a point $\boldsymbol{x} \in X$ is called an **extreme point** of X if $\boldsymbol{x}$ cannot be obtained as the convex combination of two points in X different from $\boldsymbol{x}$:

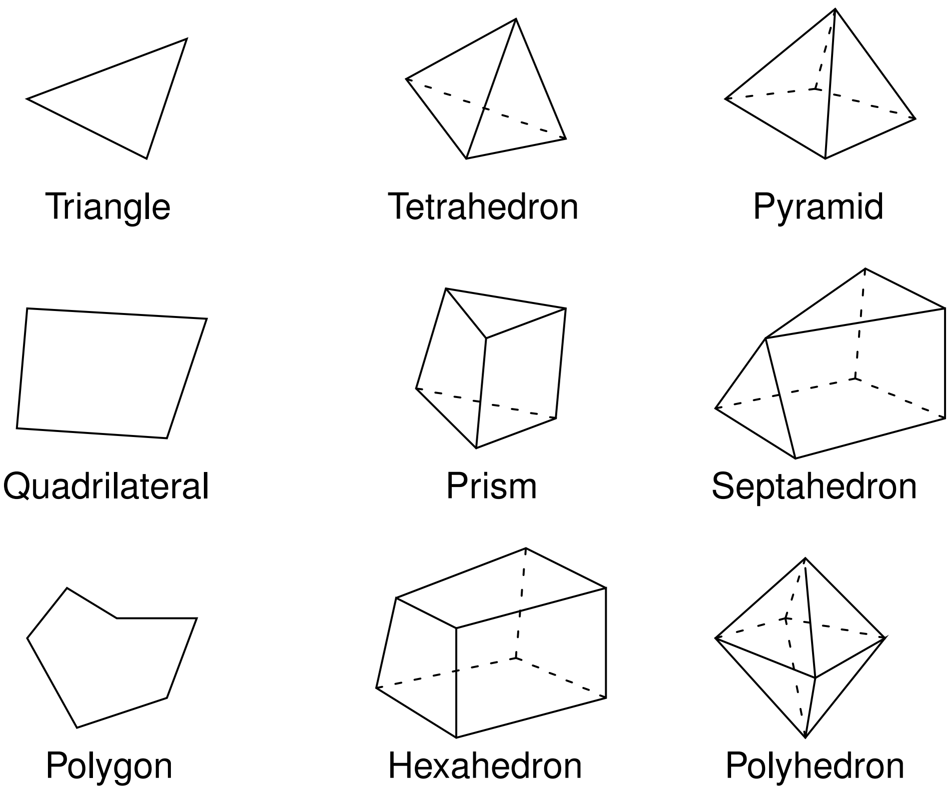
$$\boldsymbol{x} = \lambda \boldsymbol{x}_1 + (1 - \lambda)\boldsymbol{x}_2 \text{ and } 0 \leq \lambda \leq 1 \Rightarrow \boldsymbol{x_1} = \boldsymbol{x_2} = \boldsymbol{x}$$

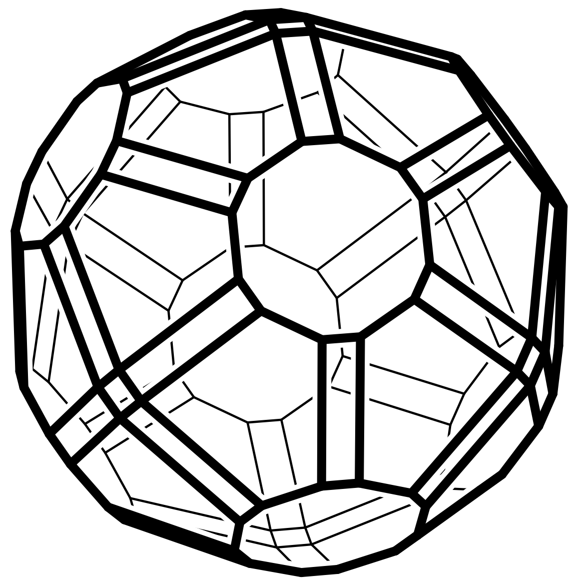


- $\boldsymbol{x}_1$ and $\boldsymbol{x}_4$ are extreme points, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ are not
- extreme points correspond to the “corner points” of a convex set

Polyhedra

- A polyhedron is a geometric object with “flat” sides





- By the word “polyhedron” we will usually mean a “convex polyhedron”

Convex Polyhedra

- **Definition 1:** the intersection of finitely many (closed) half-spaces

$$X = \{\boldsymbol{x} : \boldsymbol{a}_i \boldsymbol{x} \leq b_i, i \in \{1, \ldots, m\}\} = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$$

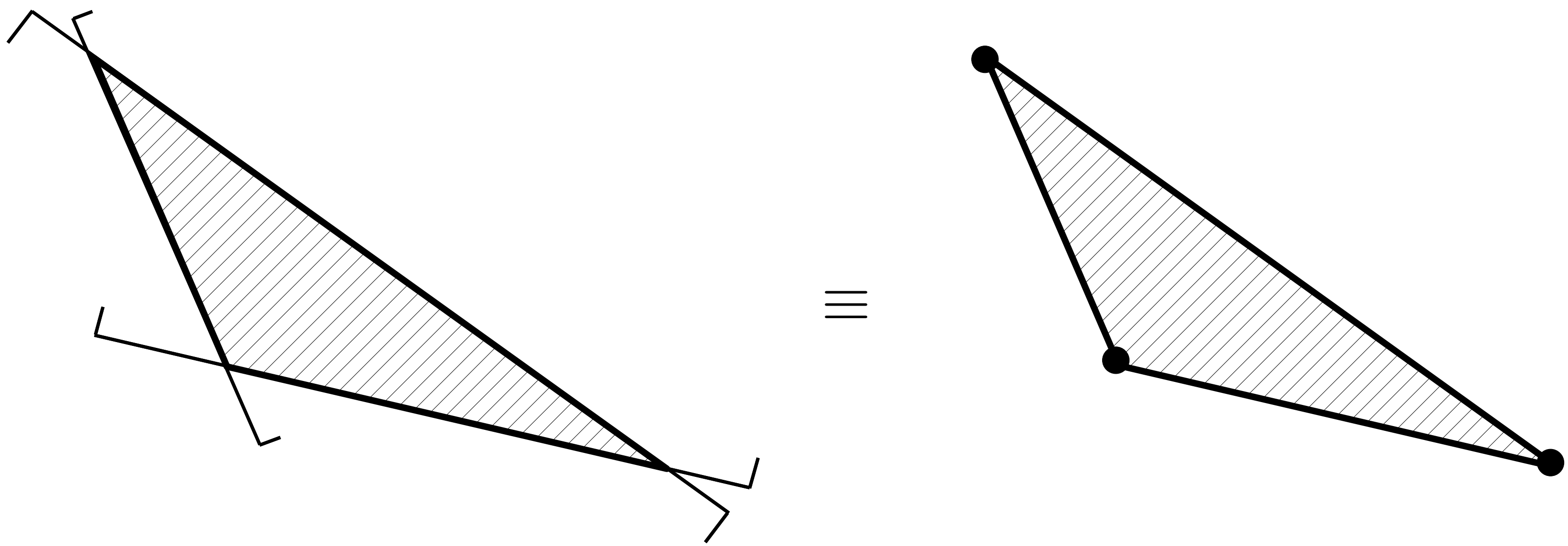
- **Corollary:** the feasible region of a linear program forms a convex polyhedron
  - canonical form: $\max\{\boldsymbol{c}^T \boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$
  - standard form: $\max\{\boldsymbol{c}^T \boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$
- **Definition 2:** convex combinations of finitely many points

$$X = \left\{ \sum_{i=1}^{n} \lambda_i \boldsymbol{x}_i : \sum_{i=1}^{n} \lambda_i = 1, \forall i \in \{1, \ldots, n\} : \lambda_i \geq 0 \right\}$$

The Minkowski-Weyl Theorem

- The Representation Theorem of Bounded Polyhedra: the two definitions are equivalent
- **The Strong Minkowski-Weyl Theorem:** if the intersection of finitely many half-spaces is bounded then it can be written as the convex combination of finitely many extreme points

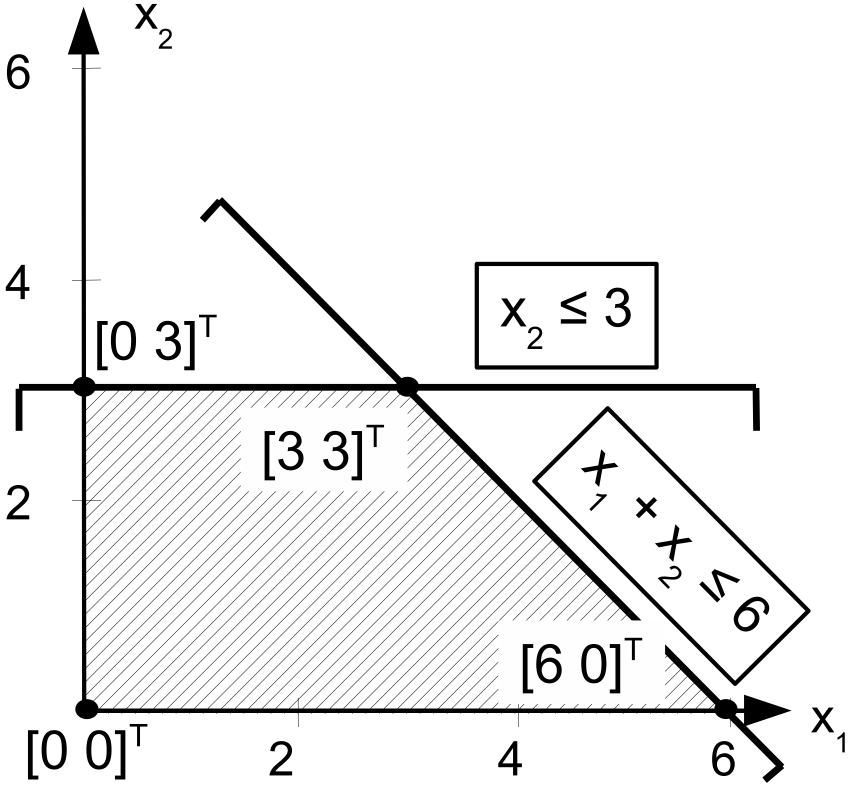
$$P = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\} \Leftrightarrow P = \text{conv}\{\boldsymbol{x}_j : 1 \leq j \leq k\}$$



Linear Programs and Extreme Points

- **The Fundamental Theorem of Linear Programming:** if the feasible region of a linear program is bounded then the at least one optimal solution is guaranteed to occur at an extreme point of the feasible region
- **Proof:** in the slide-deck, please understand and learn!
- **Bottomline:** it is not necessary to explore the entire “interior” of the feasible region, it is enough to consider a finite set of extreme points
- The simplex algorithm will do exactly that

Extreme points: Example

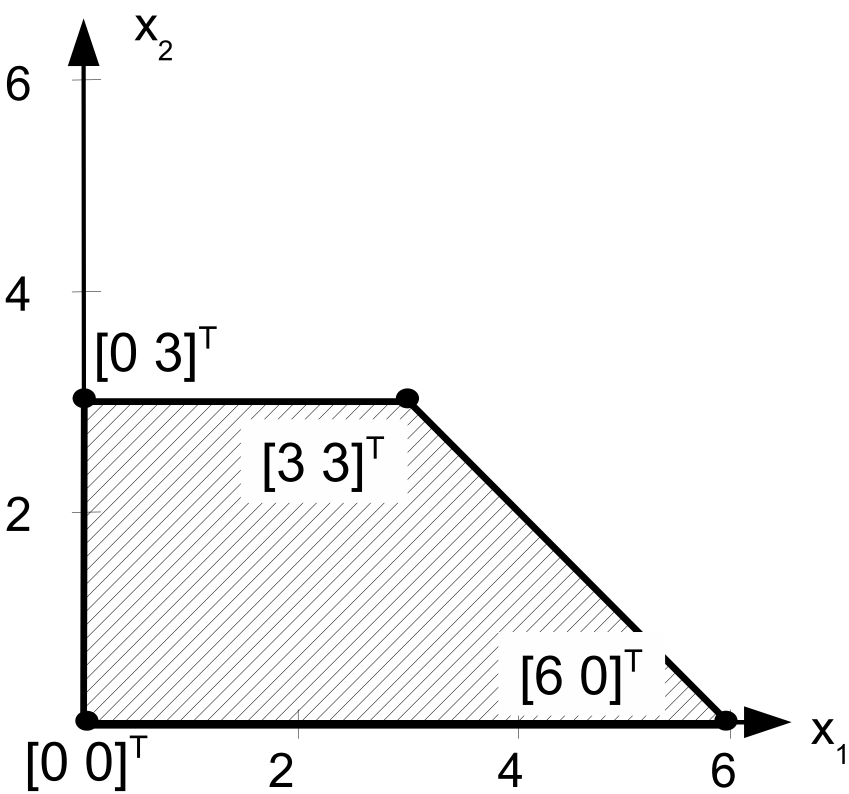


$$\begin{array}{lcccccc} \max & x_1 & + & 2x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \leq & 6 \\ & & & x_2 & \leq & 3 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

- Extreme points:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

Extreme points: Example



- Compute the objective function value $\boldsymbol{c}^T\boldsymbol{x}_j$ for each extreme point $\boldsymbol{x}_j$:

$$\boldsymbol{c}^T\boldsymbol{x}_1 = [1 \quad 2] \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

$$\boldsymbol{c}^T\boldsymbol{x}_2 = [1 \quad 2] \begin{bmatrix} 0 \\ 3 \end{bmatrix} = 6$$

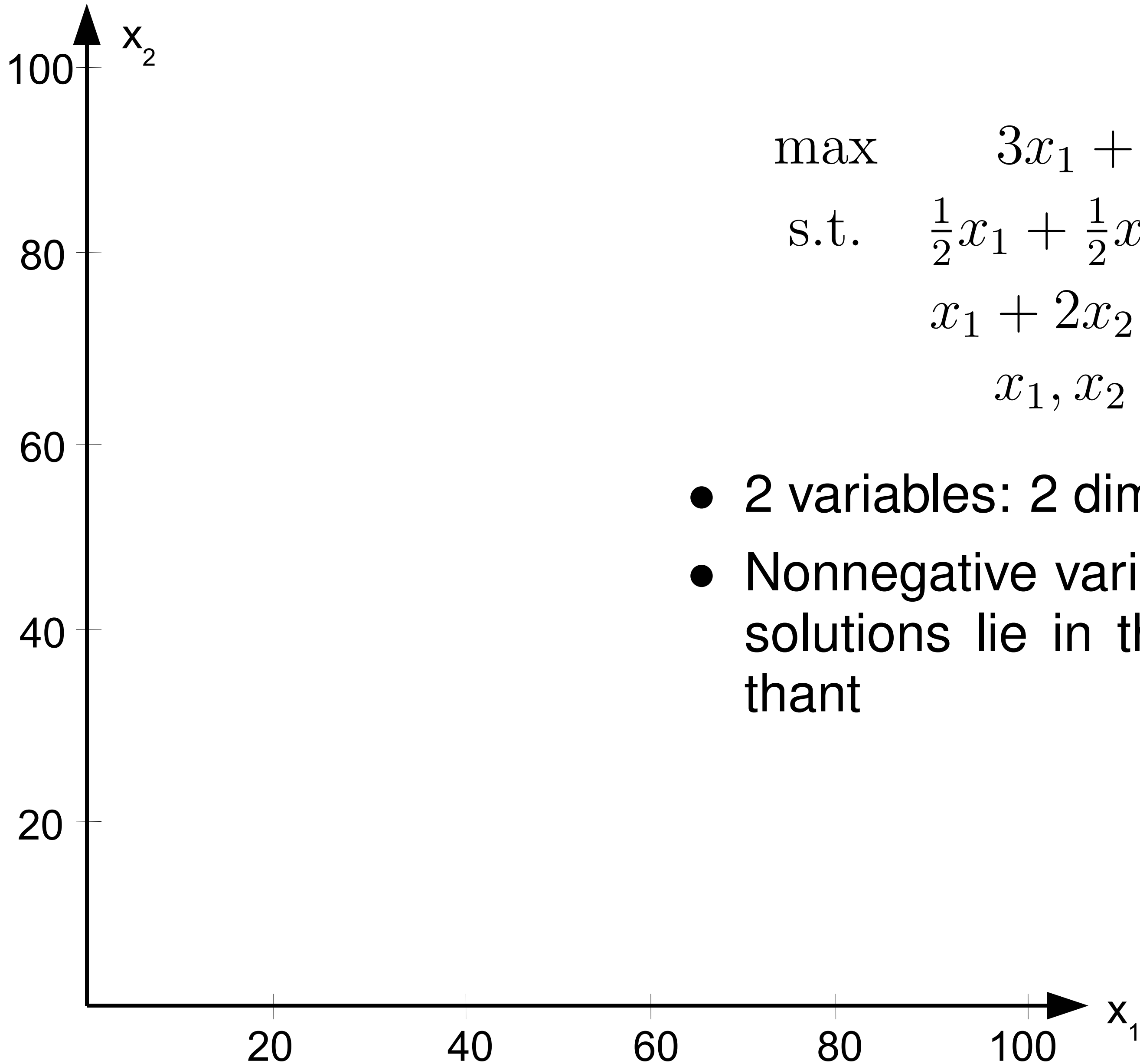
$$\boldsymbol{c}^T\boldsymbol{x}_3 = [1 \quad 2] \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \boxed{9}$$

$$\boldsymbol{c}^T\boldsymbol{x}_4 = [1 \quad 2] \begin{bmatrix} 6 \\ 0 \end{bmatrix} = 6$$

Optimal Resource Allocation Revisited

- **Exercise:** a paper mill manufactures two types of paper, standard and deluxe
  - $\frac{1}{2}$ m^3 of wood is needed to manufacture 1 m^2 of paper (both standard or deluxe)
  - producing 1 m^2 of standard paper takes 1 man-hour, whereas 1 m^2 of deluxe paper requires 2 man-hours
  - every week 40 m^3 wood and 100 man-hours of workforce is available
  - the profit is 3 thousand USD per 1 m^2 of standard paper and 4 thousand USD per 1 m^2 of deluxe paper
- **Question:** how much standard and how much deluxe paper should be produced by the paper mill per week to maximize profits?

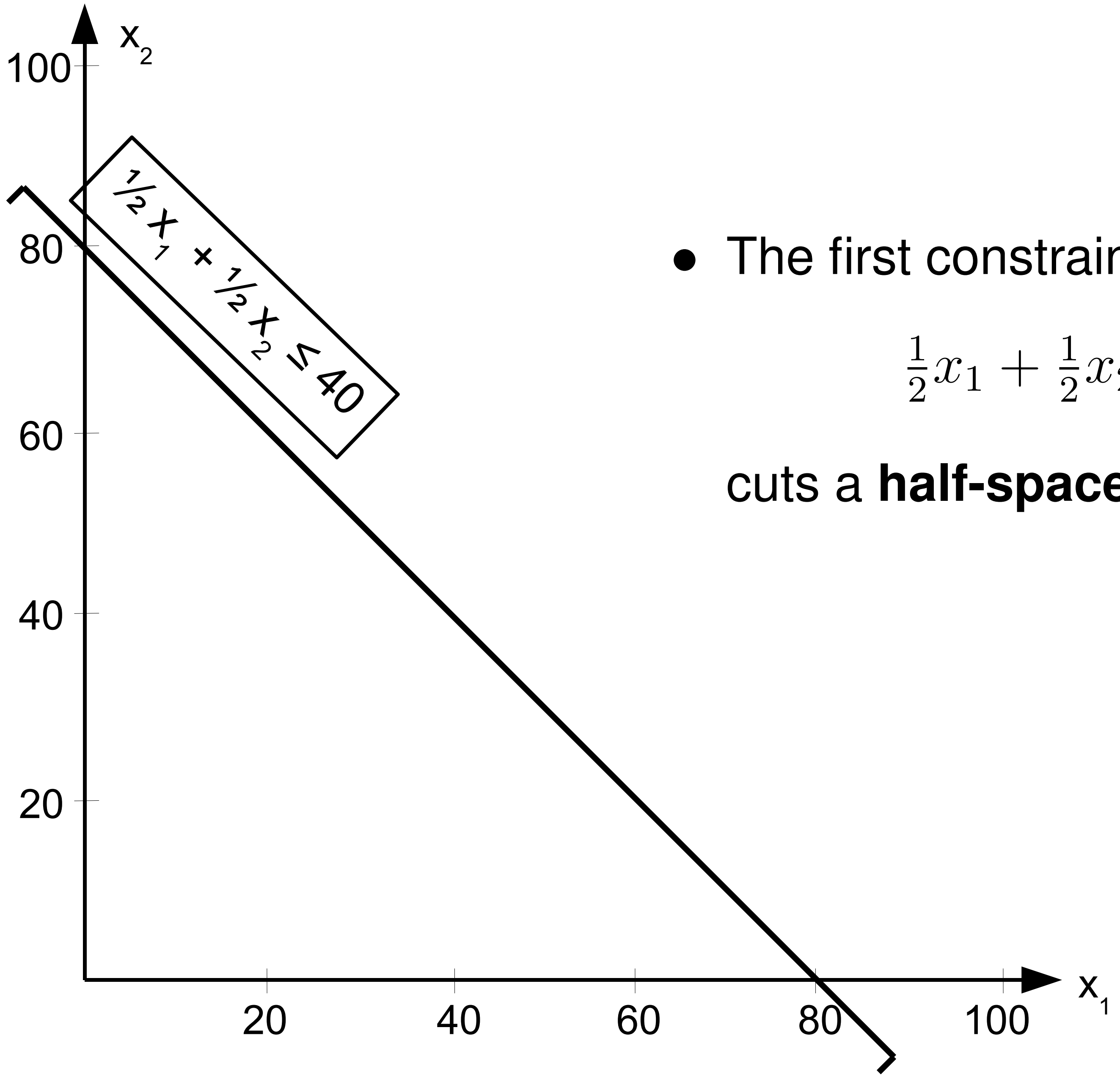
Graphical Solution



$$\begin{aligned} \max \quad & 3x_1 + 4x_2 \\ \text{s.t.} \quad & \tfrac{1}{2}x_1 + \tfrac{1}{2}x_2 \le 40 \\ & x_1 + 2x_2 \le 100 \\ & x_1, x_2 \ge 0 \end{aligned}$$

- 2 variables: 2 dimensions
- Nonnegative variables: feasible solutions lie in the positive orthant

Graphical Solution

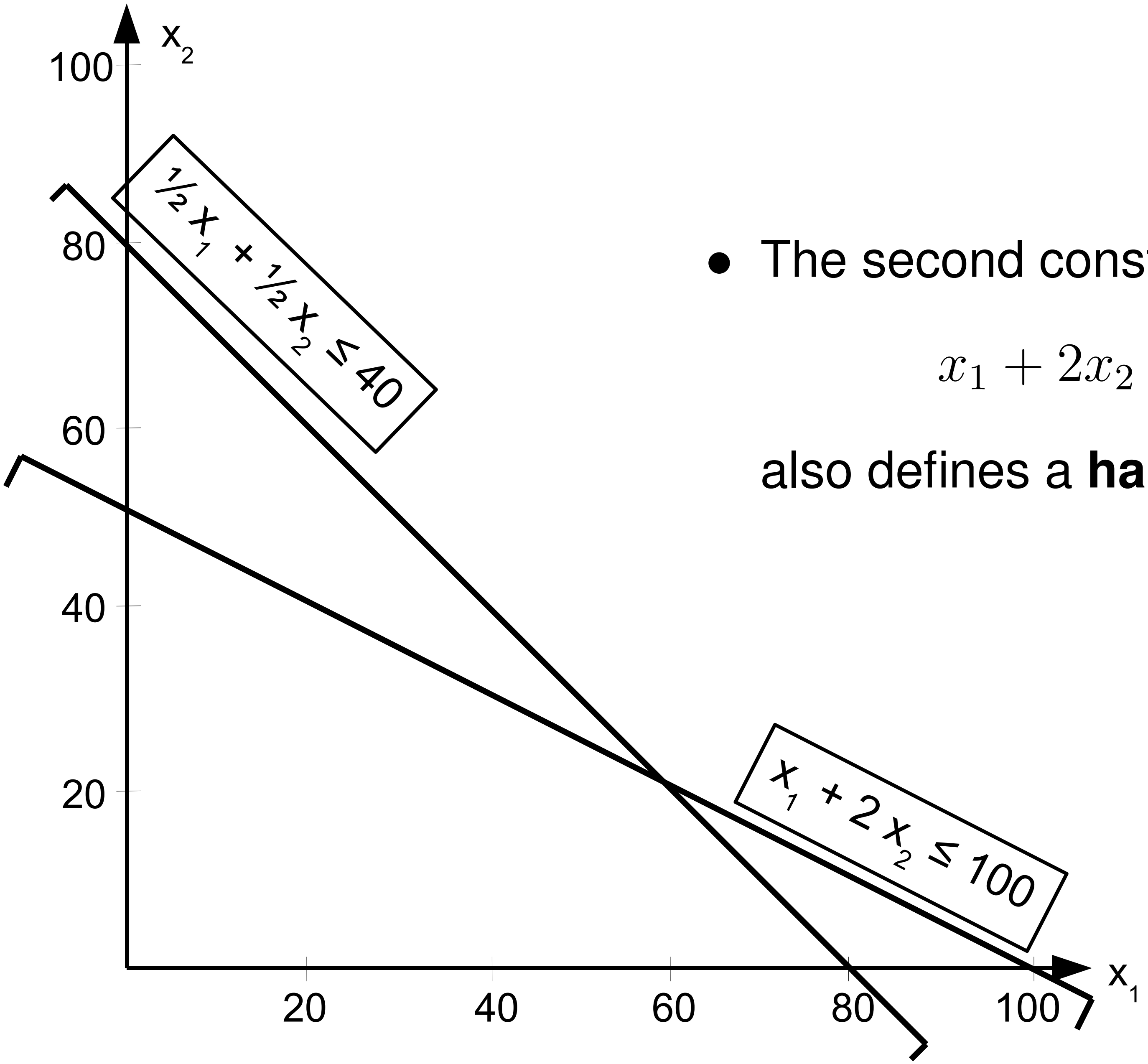


- The first constraint

$$\frac{1}{2}x_1 + \frac{1}{2}x_2 \leq 40$$

cuts a **half-space**

Graphical Solution

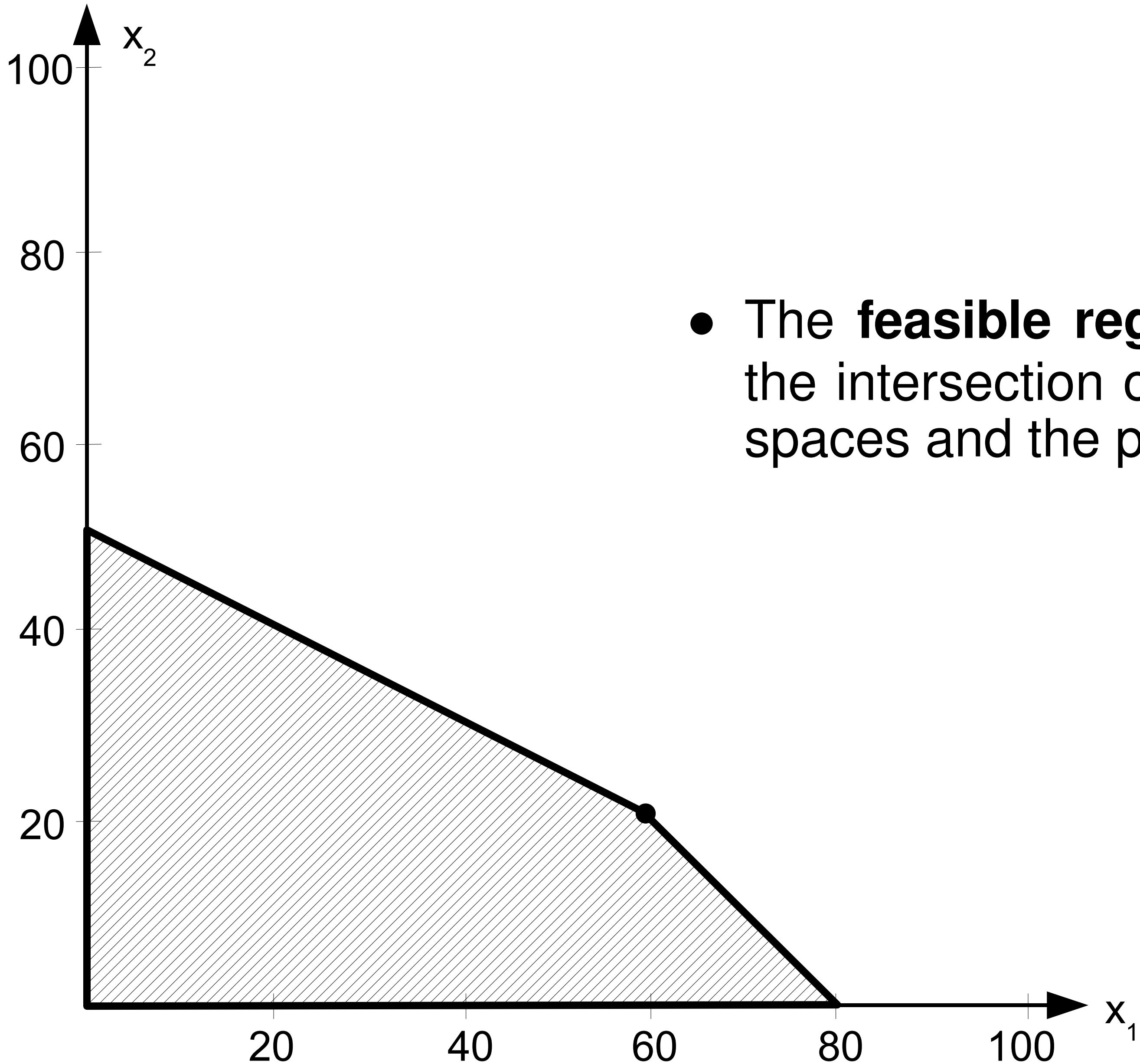


- The second constraint

$$x_1 + 2x_2 \leq 100$$

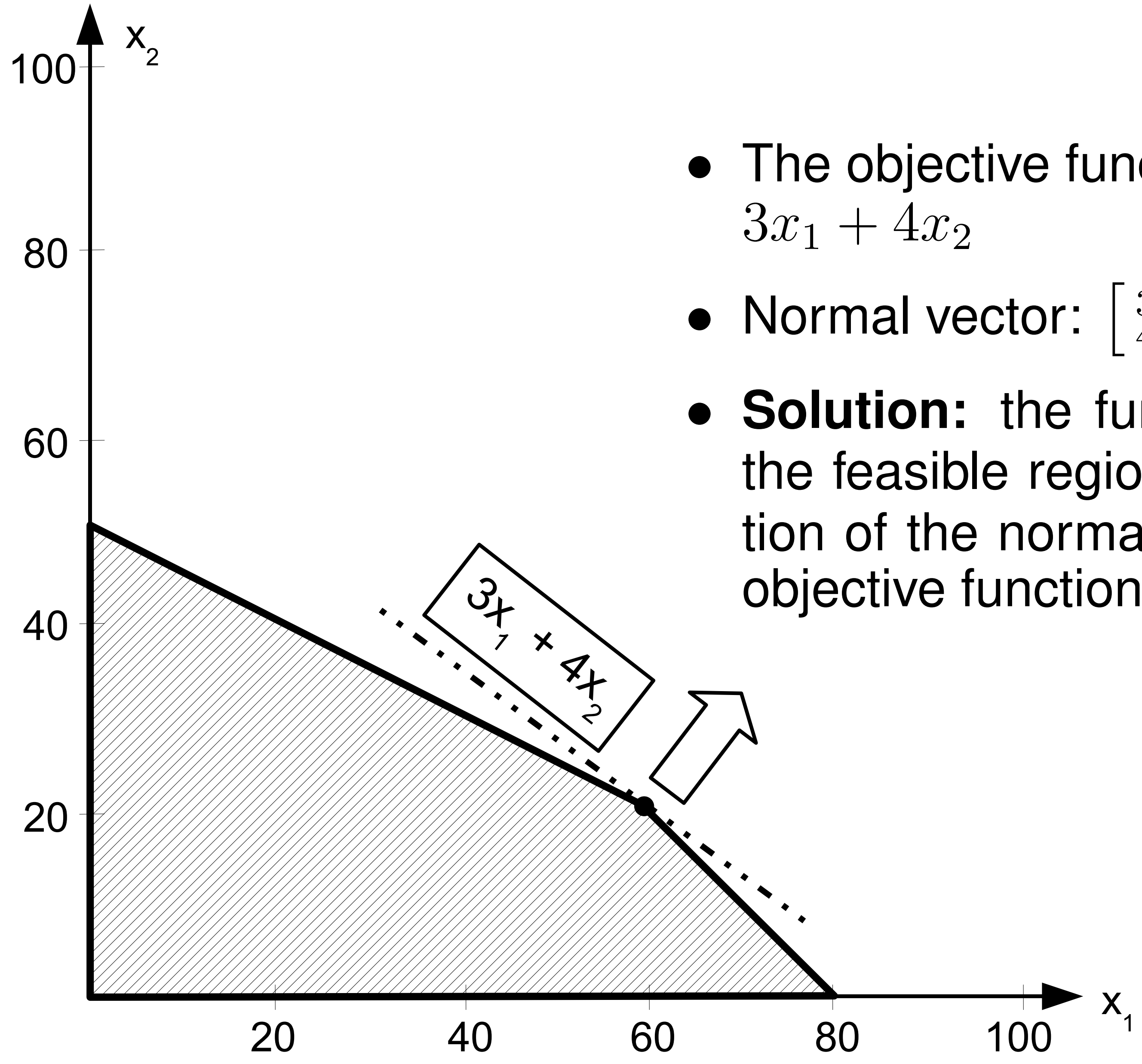
also defines a **half-space**

Graphical Solution



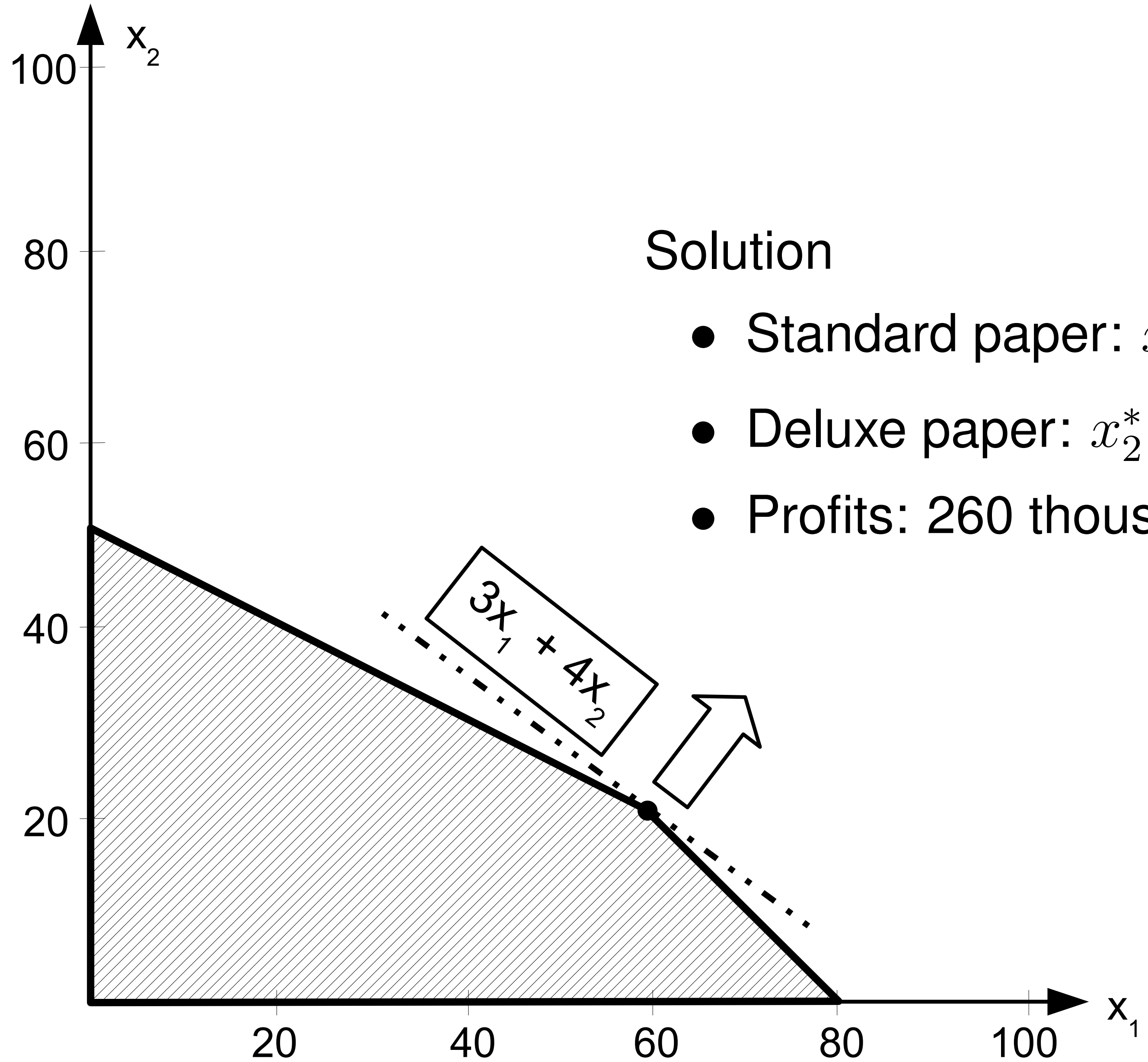
- The **feasible region** is exactly the intersection of the two half-spaces and the positive orthant

Graphical Solution



- The objective function: $3x_1 + 4x_2$
- Normal vector: $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$
- **Solution:** the furthest point of the feasible region in the direction of the normal vector of the objective function

Graphical Solution



Solution

- Standard paper: $x_1^* = 60$ m²
- Deluxe paper: $x_2^* = 20$ m²
- Profits: 260 thousand USD