Solving Linear Programs: The Basics

- Introduction to convex analysis: convexity, convex combination, hyperplanes, half-spaces, extreme points
- Convex and concave functions, the gradient, local and global optima, the fundamental theorem of convex programming
- Convex geometry: polyhedra, the Minkowski-Weyl theorem (the Representation Theorem)
- Solving linear programs using the Minkowski-Weyl theorem, the relation of optimal feasible solutions and extreme points
- Solving simple linear programs with the graphical method
- The feasible region (bounded, unbounded, empty) and optimal solutions (unique, alternative, unbounded)

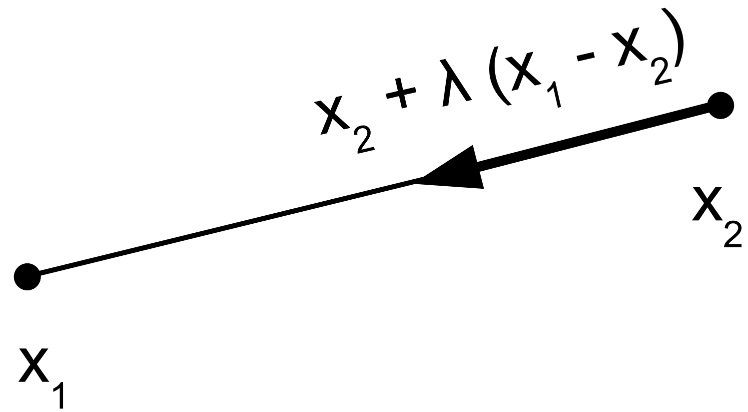
Convex Sets

- For each $0 \leq \lambda \leq 1$, the expression

$$\lambda \boldsymbol{x}_1 + (1 - \lambda)\boldsymbol{x}_2$$

  is called the **convex combination of vectors x_1 and x_2**

- Geometrically, the convex combinations of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ span the **line segment** between point $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$

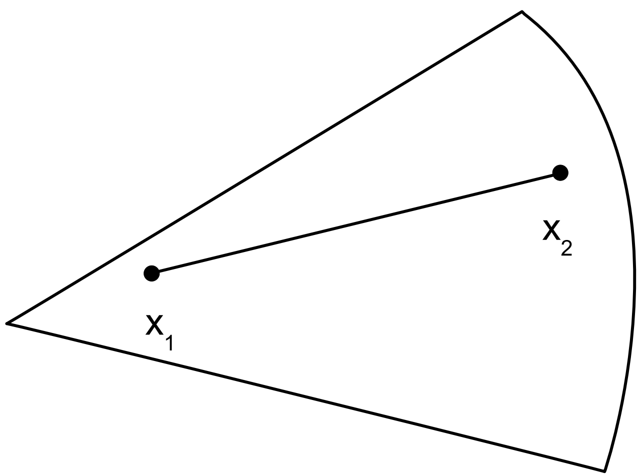


Convex Sets

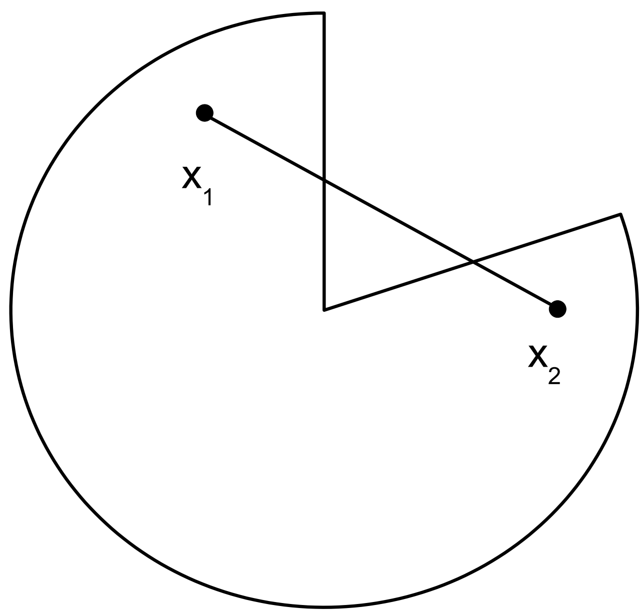
- A set $X \subset \mathbb{R}^n$ is **convex** if for each points $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in X it holds that

$$\forall \lambda \in [0, 1] : \lambda \boldsymbol{x}_1 + (1 - \lambda)\boldsymbol{x}_2 \in X$$

- In other words, X is convex if it contains all convex combinations of each of its points



Convex set



Nonconvex set

Convex Sets: Examples

- The **convex combinations of k points $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k$:**

$$X = \left\{ \sum_{i=1}^{k} \lambda_i \boldsymbol{x}_i : \sum_{i=1}^{k} \lambda_i = 1, \forall i \in \{1, \ldots, k\} : \lambda_i \geq 0 \right\}$$

$$X = \operatorname{conv}\{\boldsymbol{x}_i : 1 \leq i \leq k\}$$

- The 3-sphere: $X = \{[x, y, z] : x^2 + y^2 + x^2 \leq 1\}$
- Vector space: $X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\}$
- Affine space (translated vector space): $X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$
- **Feasible region of a linear program:**

$$X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$$

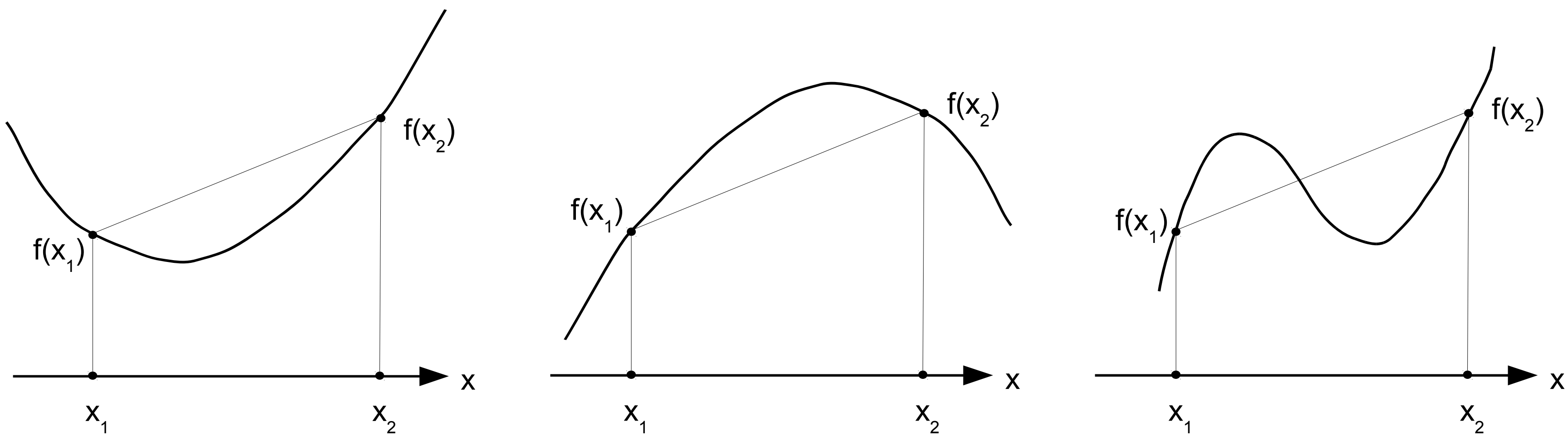
$$X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$$

Convex and Concave Functions

- A function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is **convex** on a convex set $X \subseteq \mathbb{R}^n$ if for each $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in X:

$$f(\lambda \boldsymbol{x}_1 + (1-\lambda)\boldsymbol{x}_2) \leq \lambda f(\boldsymbol{x}_1) + (1-\lambda) f(\boldsymbol{x}_2) \quad \forall \lambda \in [0,1]$$

- The line segment between any two points $f(\boldsymbol{x}_1)$ and $f(\boldsymbol{x}_2)$ on the graph of the function lies above or on the graph
- Function f is **concave** if $(-f)$ is convex



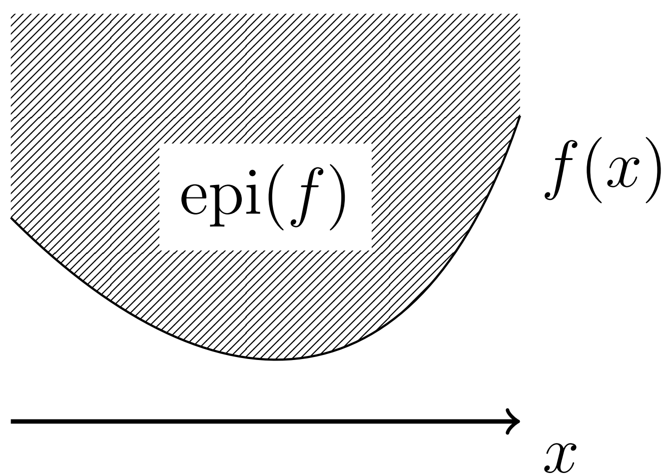
(a) convex (b) concave (c) neither

Convex and Concave Functions

- The set of points "above" the graph of some function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is called the **epigraph** of f

$$\mathrm{epi}(f) = \{(\boldsymbol{x}, y) : \boldsymbol{x} \in \mathbb{R}^n, y \in \mathbb{R}, y \geq f(\boldsymbol{x})\}$$

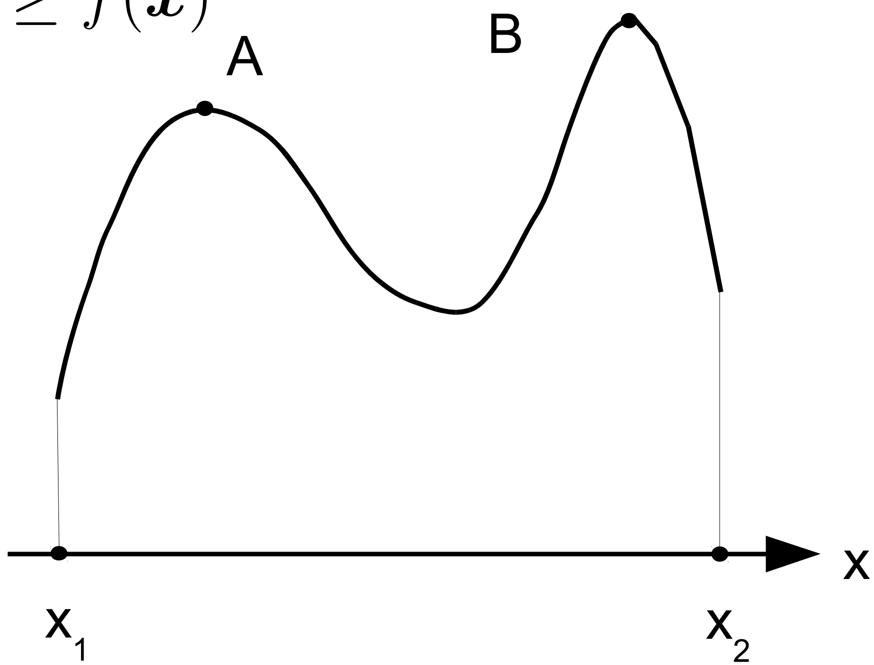
- Simple convexity check: f is convex on set $\mathbb{R}^n$ if and only if $\mathrm{epi}(f)$ is convex



Optimization on a Convex Set

- Given function $f : \mathbb{R}^n \mapsto \mathbb{R}$ and set X, solve the generic optimization problem $\max f(\boldsymbol{x}) : \boldsymbol{x} \in X$
- Some $\bar{\boldsymbol{x}} \in X$ is a **global optimal solution** (or global optimum) if for each $\boldsymbol{x} \in X$: $f(\bar{\boldsymbol{x}}) \geq f(\boldsymbol{x})$
- An $\bar{\boldsymbol{x}} \in X$ is a **local optimum** if there is a neighborhood $N_\epsilon(\bar{\boldsymbol{x}})$ of $\bar{\boldsymbol{x}}$ (an open ball of radius $\epsilon > 0$ with centre $\bar{\boldsymbol{x}}$) so that $\forall \boldsymbol{x} \in N_\epsilon(\bar{\boldsymbol{x}}) \cap X$: $f(\bar{\boldsymbol{x}}) \geq f(\boldsymbol{x})$

point A is a local optimum and point B is a global optimum on the closed interval $[x_1, x_2]$



Optimization on a Convex Set

- **Theorem:** Let X be a nonempty **convex set** in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \to \mathbb{R}$ be a **concave function** on X. Consider the optimization problem $\max f(\boldsymbol{x}) : \boldsymbol{x} \in X$. Then, if $\bar{\boldsymbol{x}} \in X$ is a local optimal solution then it is also a global optimum
- **Proof:** $\bar{\boldsymbol{x}} \in X$ is a local optimum, therefore there is nonempty neighborhood $N_\epsilon(\bar{\boldsymbol{x}})$, $\epsilon > 0$ so that

$$\forall \boldsymbol{x} \in N_\epsilon(\bar{\boldsymbol{x}}) \cap X : f(\bar{\boldsymbol{x}}) \geq f(\boldsymbol{x})$$

- Suppose that $\bar{\boldsymbol{x}}$ is not a global optimal solution and therefore there exists $\hat{\boldsymbol{x}} \in X : f(\hat{\boldsymbol{x}}) > f(\bar{\boldsymbol{x}})$
- Since X is convex it contains all convex combinations of $\hat{\boldsymbol{x}}$ and $\bar{\boldsymbol{x}}$ (i.e., the line segment between $\hat{\boldsymbol{x}}$ and $\bar{\boldsymbol{x}}$)
- Also, since $N_\epsilon(\bar{\boldsymbol{x}})$ is nonempty it contains points different from $\bar{\boldsymbol{x}}$ that are also on the line segment between $\hat{\boldsymbol{x}}$ and $\bar{\boldsymbol{x}}$

Optimization on a Convex Set

- Since f is concave, the graph of f lies above, or on the line segment joining $\hat{\boldsymbol{x}}$ and $\bar{\boldsymbol{x}}$

$$\begin{aligned}\forall\lambda \in (0,1] : f(\lambda\hat{\boldsymbol{x}} + (1-\lambda)\bar{\boldsymbol{x}}) &\geq \lambda f(\hat{\boldsymbol{x}}) + (1-\lambda) f(\bar{\boldsymbol{x}}) \\ &\stackrel{*}{>} \lambda f(\bar{\boldsymbol{x}}) + (1-\lambda) f(\bar{\boldsymbol{x}}) = f(\bar{\boldsymbol{x}})\end{aligned}$$

- The strict inequality (marked with * above) comes from the assumption that $\bar{\boldsymbol{x}}$ is not a global optimum: $f(\hat{\boldsymbol{x}}) > f(\bar{\boldsymbol{x}})$
- Consequently, all neighborhoods of $\bar{\boldsymbol{x}}$ contain feasible points $\boldsymbol{x} \neq \bar{\boldsymbol{x}}$ for which $f(\boldsymbol{x}) > f(\bar{\boldsymbol{x}})$, which contradicts our assumption that $\bar{\boldsymbol{x}}$ is a local optimal solution $\square$
- **Corollary:** Let X be a nonempty **convex set** in $\mathbb{R}^n$, let $f : \mathbb{R}^n \to \mathbb{R}$ be a **convex function** on X, and let $\bar{\boldsymbol{x}} \in X$ be a local optimal solution to the optimization problem $\min f(\boldsymbol{x}) : \boldsymbol{x} \in X$. Then, $\bar{\boldsymbol{x}}$ is also a global optimal solution

The Gradient

- Given a set $X \subseteq \mathbb{R}^n$ and function $f : \mathbb{R}^n \mapsto \mathbb{R}$. We call f **differentiable** at a point $\bar{\boldsymbol{x}} \in X$ if there exits $\nabla f(\bar{\boldsymbol{x}})$ **gradient vector** and function $\alpha : \mathbb{R}^n \mapsto \mathbb{R}$ so that for all $\boldsymbol{x} \in X$:

$$f(\boldsymbol{x}) = f(\bar{\boldsymbol{x}}) + \nabla f(\bar{\boldsymbol{x}})^T(\boldsymbol{x} - \bar{\boldsymbol{x}}) + \|\boldsymbol{x} - \bar{\boldsymbol{x}}\|\alpha(\bar{\boldsymbol{x}}, \boldsymbol{x} - \bar{\boldsymbol{x}}) \ \ ,$$

  where $\lim_{\boldsymbol{x} \to \bar{\boldsymbol{x}}} \alpha(\bar{\boldsymbol{x}}, \boldsymbol{x} - \bar{\boldsymbol{x}}) = 0$

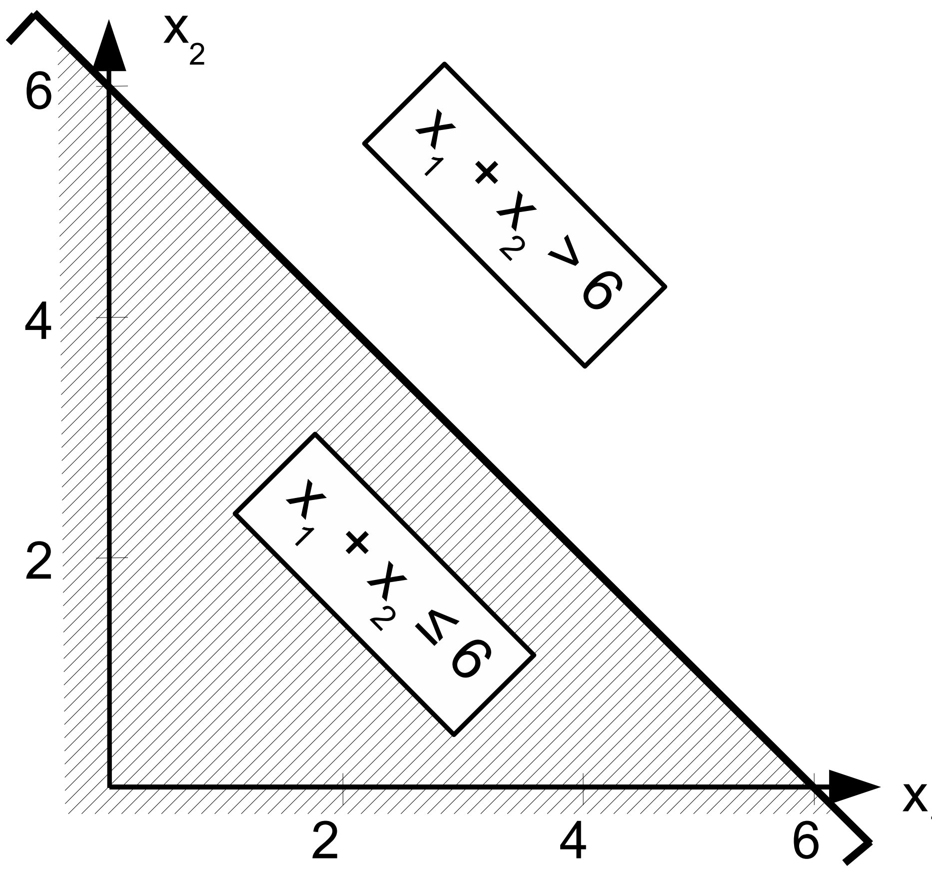
- "Linearization" by approximating with the first-order Taylor series: $f(\boldsymbol{x}) \approx f(\bar{\boldsymbol{x}}) + \nabla f(\bar{\boldsymbol{x}})^T(\boldsymbol{x} - \bar{\boldsymbol{x}})$

- If f is differentiable then the gradient is well-defined and can be written as the vector of the partial derivatives:

$$\nabla f(\boldsymbol{x})^T = \begin{bmatrix} \frac{\partial f(\boldsymbol{x})}{\partial x_1} & \frac{\partial f(\boldsymbol{x})}{\partial x_2} & \dots & \frac{\partial f(\boldsymbol{x})}{\partial x_n} \end{bmatrix}$$

Hyperplanes and Half-spaces

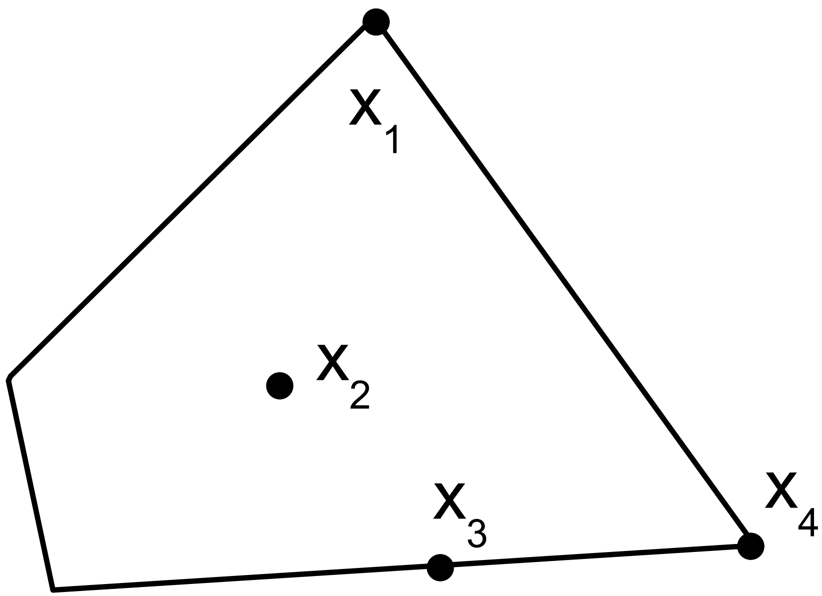
- **Hyperplane:** all $\boldsymbol{x} \in \mathbb{R}^n$ satisfying the equation $\boldsymbol{a}^T\boldsymbol{x} = b$ for some $\boldsymbol{a}^T$ row n-vector (the **normal vector**) and scalar b
- The hyperplane $X = \{\boldsymbol{x} : \boldsymbol{a}^T\boldsymbol{x} = b\}$ divides the space $\mathbb{R}^n$ into two **half-spaces**
  - "lower" half-space: $\{\boldsymbol{x} : \boldsymbol{a}^T\boldsymbol{x} \leq b\}$
  - "upper" half-space: $\{\boldsymbol{x} : \boldsymbol{a}^T\boldsymbol{x} > b\}$
- Hyperplanes and half-spaces are convex



Extreme Points

- Given a convex set X, a point $\boldsymbol{x} \in X$ is called an **extreme point** of X if $\boldsymbol{x}$ cannot be obtained as the convex combination of two points in X different from $\boldsymbol{x}$:

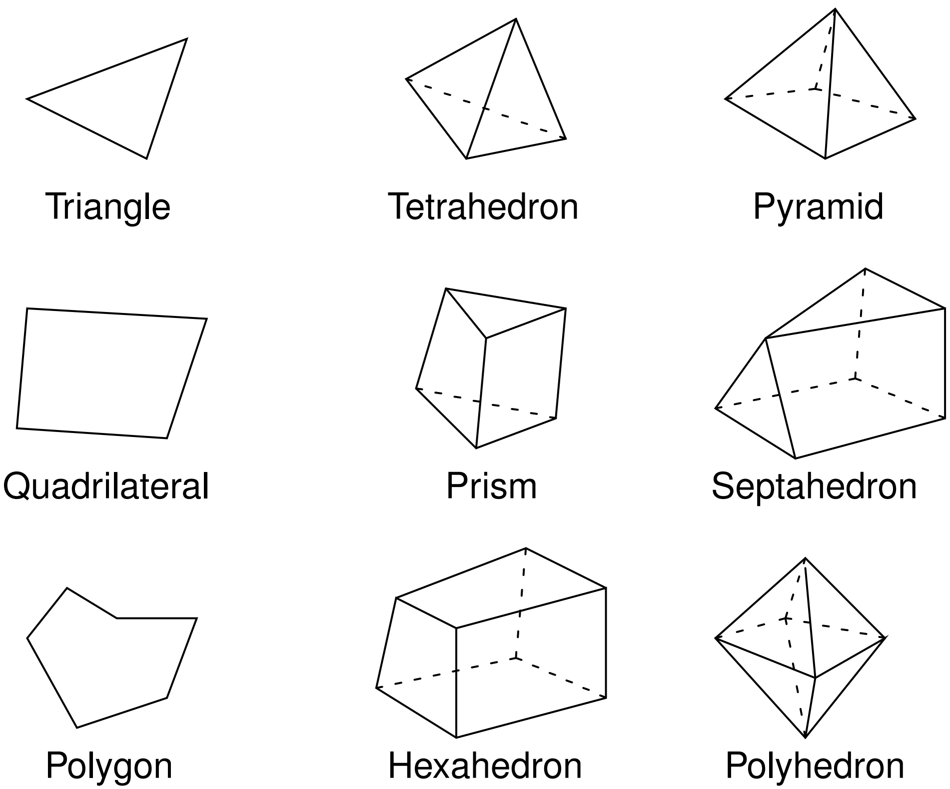
$$\boldsymbol{x} = \lambda \boldsymbol{x}_1 + (1 - \lambda)\boldsymbol{x}_2 \text{ and } 0 \leq \lambda \leq 1 \Rightarrow \boldsymbol{x_1} = \boldsymbol{x_2} = \boldsymbol{x}$$

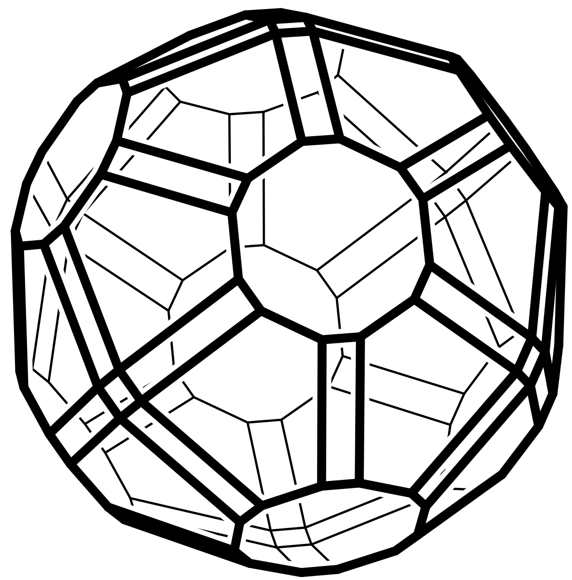


- $\boldsymbol{x}_1$ and $\boldsymbol{x}_4$ are extreme points, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$ are not
- extreme points correspond to the “corner points” of a convex set

Polyhedra

- A polyhedron is a geometric object with “flat” sides





- By the word “polyhedron” we will usually mean a “convex polyhedron”

Convex Polyhedra

- **Definition 1:** the intersection of finitely many (closed) half-spaces

$$X = \{\boldsymbol{x} : \boldsymbol{a}_i \boldsymbol{x} \leq b_i, i \in \{1, \ldots, m\}\} = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\}$$

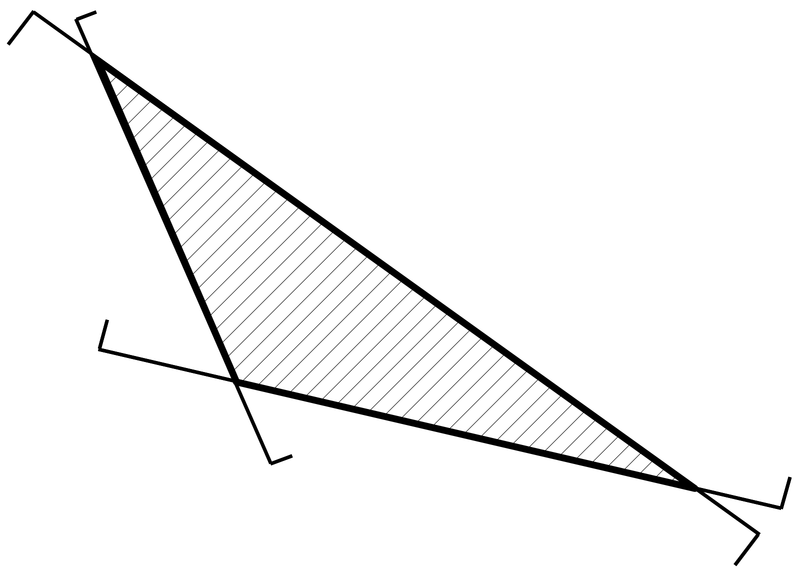
- **Corollary:** the feasible region of a linear program forms a convex polyhedron
  - canonical form: $\max\{\boldsymbol{c}^T \boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$
  - standard form: $\max\{\boldsymbol{c}^T \boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$
- **Definition 2:** convex combinations of finitely many points

$$X = \left\{ \sum_{i=1}^{n} \lambda_i \boldsymbol{x}_i : \sum_{i=1}^{n} \lambda_i = 1, \forall i \in \{1, \ldots, n\} : \lambda_i \geq 0 \right\}$$

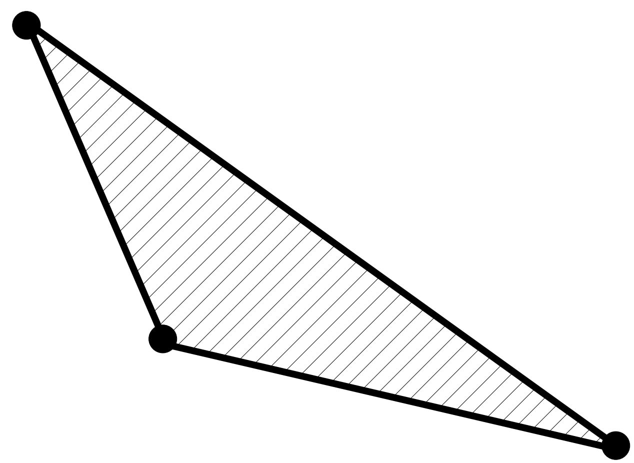
The Minkowski-Weyl Theorem

- The Representation Theorem of Bounded Polyhedra: the two definitions are equivalent
- **The Strong Minkowski-Weyl Theorem:** if the intersection of finitely many half-spaces is bounded then it can be written as the convex combination of finitely many extreme points

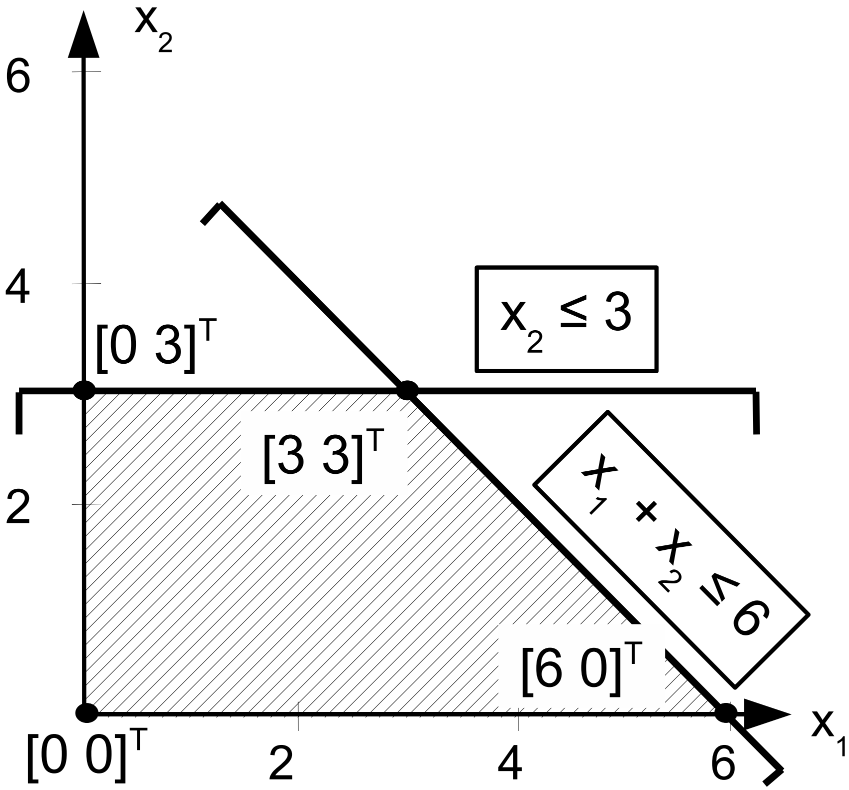
$$P = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}\} \Leftrightarrow P = \mathrm{conv}\{\boldsymbol{x}_j : 1 \leq j \leq k\}$$



$$\equiv$$



The Minkowski-Weyl Theorem: Example



Extreme points:

$$\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\3\end{bmatrix}, \begin{bmatrix}3\\3\end{bmatrix}, \begin{bmatrix}6\\0\end{bmatrix}$$

- Half-space intersection:

$$X = \{x_1, x_2 : \quad \begin{aligned} x_1 + x_2 &\leq 6 \\ x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned} \quad \}$$

- Convex combination of extreme points:

$$X = \mathrm{conv}\left(\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\3\end{bmatrix}, \begin{bmatrix}3\\3\end{bmatrix}, \begin{bmatrix}6\\0\end{bmatrix} \right)$$

Linear Programs and Extreme Points

- **The Fundamental Theorem of Linear Programming:** if the feasible region of a linear program is bounded then the at least one optimal solution is guaranteed to occur at an extreme point of the feasible region
- **Proof:** given the below linear program

$$\begin{aligned} \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

- May also be cast in the form $\max\{\boldsymbol{c}^T \boldsymbol{x} : \boldsymbol{A}'\boldsymbol{x} \leq \boldsymbol{b}'\}$

$$\begin{aligned} \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\ & -\boldsymbol{A}\boldsymbol{x} \leq -\boldsymbol{b} \\ & -\boldsymbol{x} \leq \boldsymbol{0} \end{aligned}$$

Lineáris Programs and Extreme Points

- The feasible region $X = \{\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ of any linear program is a polyhedron
- Suppose now that X is bounded and let the extreme points of X be $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, . . ., $\boldsymbol{x}_k$
- Using the strong Minkowski-Weyl theorem, X can be written as the convex combination of its extreme points

$$\forall \boldsymbol{x} \in X : \qquad \boldsymbol{x} = \sum_{j=1}^{k} \lambda_j \boldsymbol{x}_j$$

$$\sum_{j=1}^{k} \lambda_j = 1$$

$$\lambda_j \geq 0$$

Lineáris Programs and Extreme Points

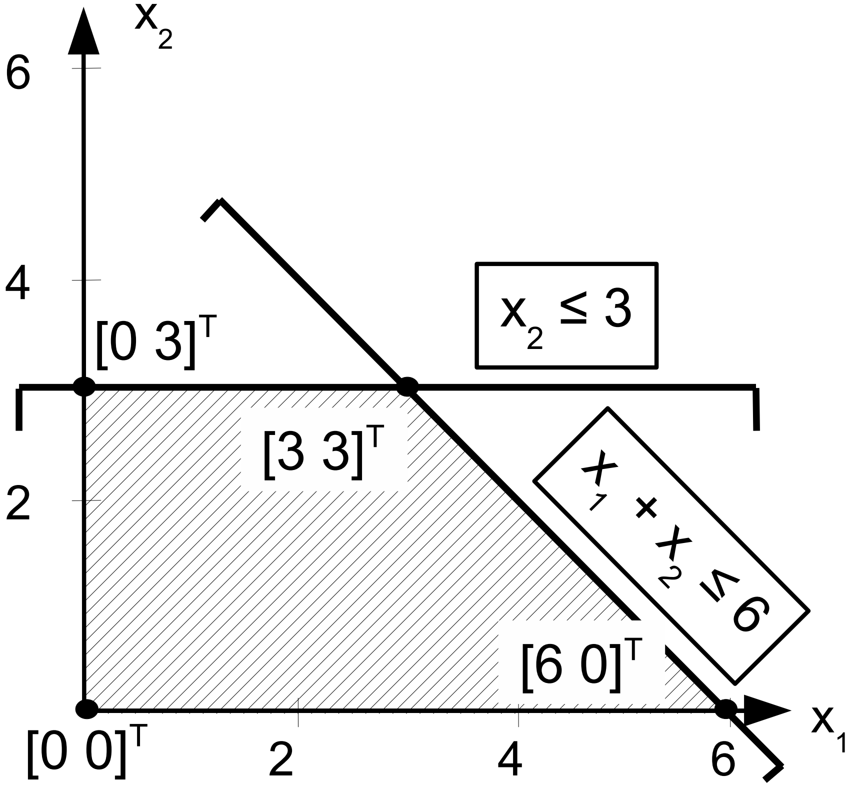
- Substitute the above $\boldsymbol{x}$ into the linear program
- Observe that $\boldsymbol{x}$ automatically satisfies the constraints!
- The objective function: $\max \boldsymbol{c}^T(\sum_{j=1}^k \lambda_j \boldsymbol{x}_j) = \max \sum_{j=1}^k (\boldsymbol{c}^T \boldsymbol{x}_j)\lambda_j$
- We have obtained another linear program in which the variables are now the scalars λ_j

$$
\begin{aligned}
\max \ & \sum_{j=1}^{k} (\boldsymbol{c}^T \boldsymbol{x}_j)\lambda_j \\
\text{s.t.} \ & \sum_{j=1}^{k} \lambda_j = 1 \\
& \lambda_j \geq 0 \qquad \forall j \in \{1, \ldots, k\}
\end{aligned}
$$

Lineáris Programs and Extreme Points

- Our goal is to obtain the maximum of the linear program $\sum_{j=1}^{k}(\boldsymbol{c}^T\boldsymbol{x}_j)\lambda_j : \sum_{j=1}^{k}\lambda_j = 1, \lambda_j \geq 0$
- It is easy to see that it is enough to find the extreme point for which $\boldsymbol{c}^T\boldsymbol{x}_j$ is maximal, let this extreme point be $\boldsymbol{x}_p$
- Then, $\lambda_p = 1$, $\lambda_j = 0 : j \neq p$ solves the new linear program and the optimal objective function value is $\boldsymbol{c}^T\boldsymbol{x}_p$. $\square$
- We have proved only for the bounded case, but the theorem also holds for the linear programs with unbounded feasible region
- In general a linear program cannot be written in the equivalent using extreme points (which would then be easy to solve), since the number of extreme points is exponential
- In lower dimensions it can: graphical method

Extreme points: Example

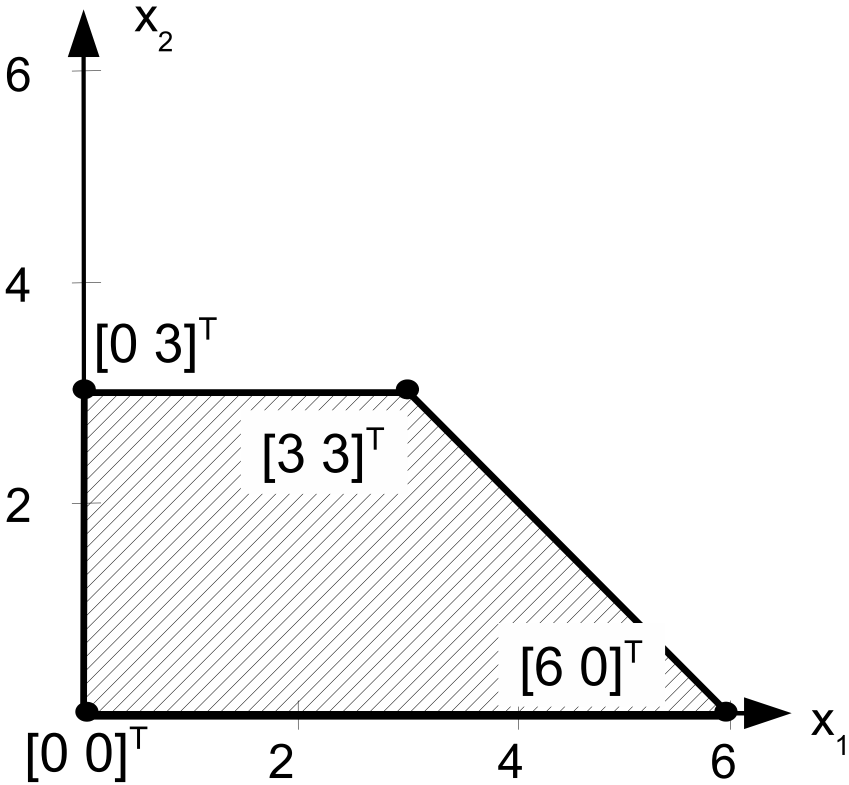


$$\begin{array}{lcccccc} \max & x_1 & + & 2x_2 & & \\ \text{s.t.} & x_1 & + & x_2 & \leq & 6 \\ & & & x_2 & \leq & 3 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

- Extreme points:

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

Extreme points: Example



- Compute the objective function value $\boldsymbol{c}^T\boldsymbol{x}_j$ for each extreme point $\boldsymbol{x}_j$:

$$\boldsymbol{c}^T\boldsymbol{x}_1 = [1 \quad 2]\begin{bmatrix}0\\0\end{bmatrix} = \quad 0$$

$$\boldsymbol{c}^T\boldsymbol{x}_2 = [1 \quad 2]\begin{bmatrix}0\\3\end{bmatrix} = \quad 6$$

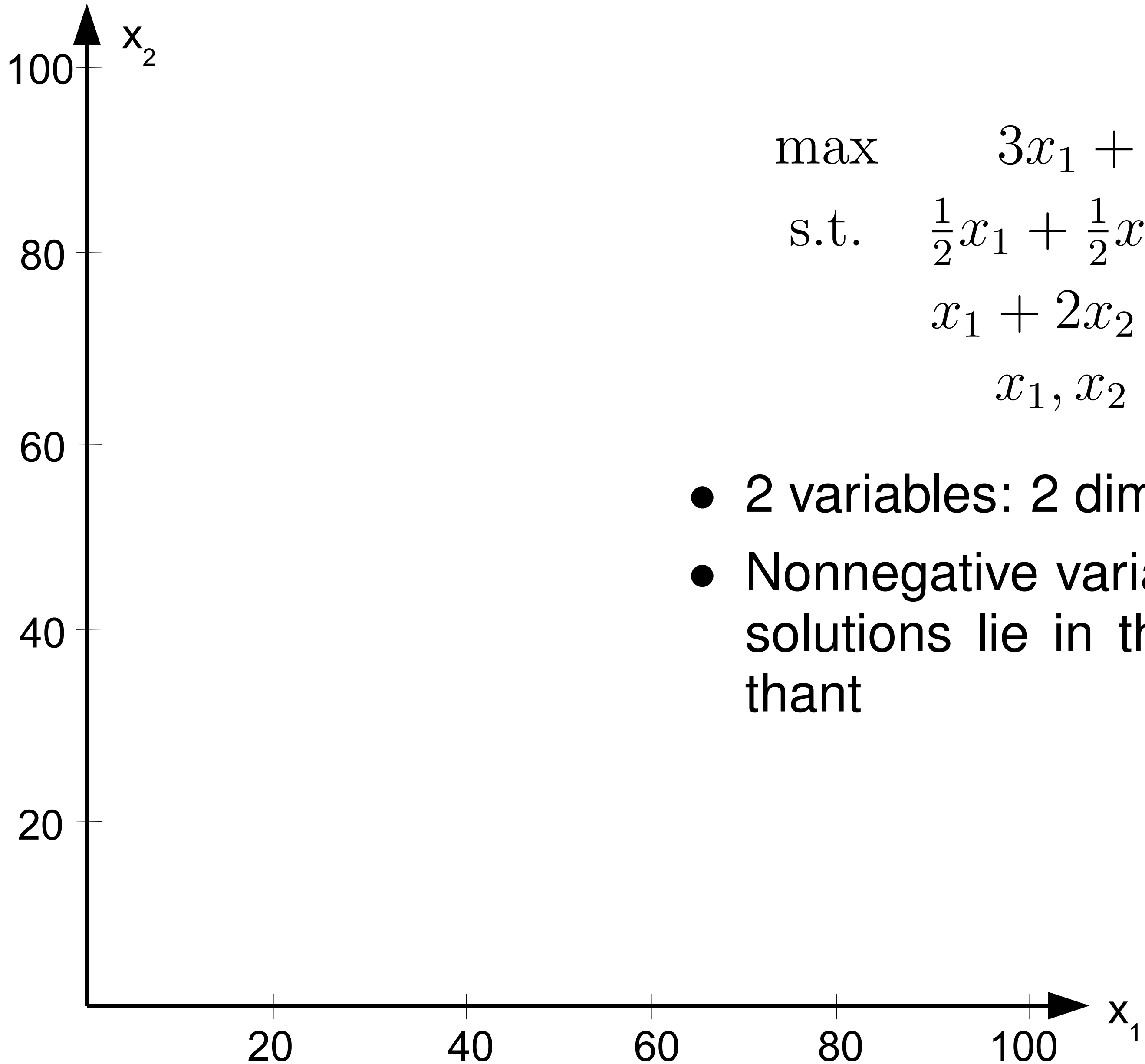
$$\boldsymbol{c}^T\boldsymbol{x}_3 = [1 \quad 2]\begin{bmatrix}3\\3\end{bmatrix} = \boxed{\ 9}$$

$$\boldsymbol{c}^T\boldsymbol{x}_4 = [1 \quad 2]\begin{bmatrix}6\\0\end{bmatrix} = \quad 6$$

Optimal Resource Allocation Revisited

- **Exercise:** a paper mill manufactures two types of paper, standard and deluxe
  - $\frac{1}{2}$ m^3 of wood is needed to manufacture 1 m^2 of paper (both standard or deluxe)
  - producing 1 m^2 of standard paper takes 1 man-hour, whereas 1 m^2 of deluxe paper requires 2 man-hours
  - every week 40 m^3 wood and 100 man-hours of workforce is available
  - the profit is 3 thousand USD per 1 m^2 of standard paper and 4 thousand USD per 1 m^2 of deluxe paper
- **Question:** how much standard and how much deluxe paper should be produced by the paper mill per week to maximize profits?

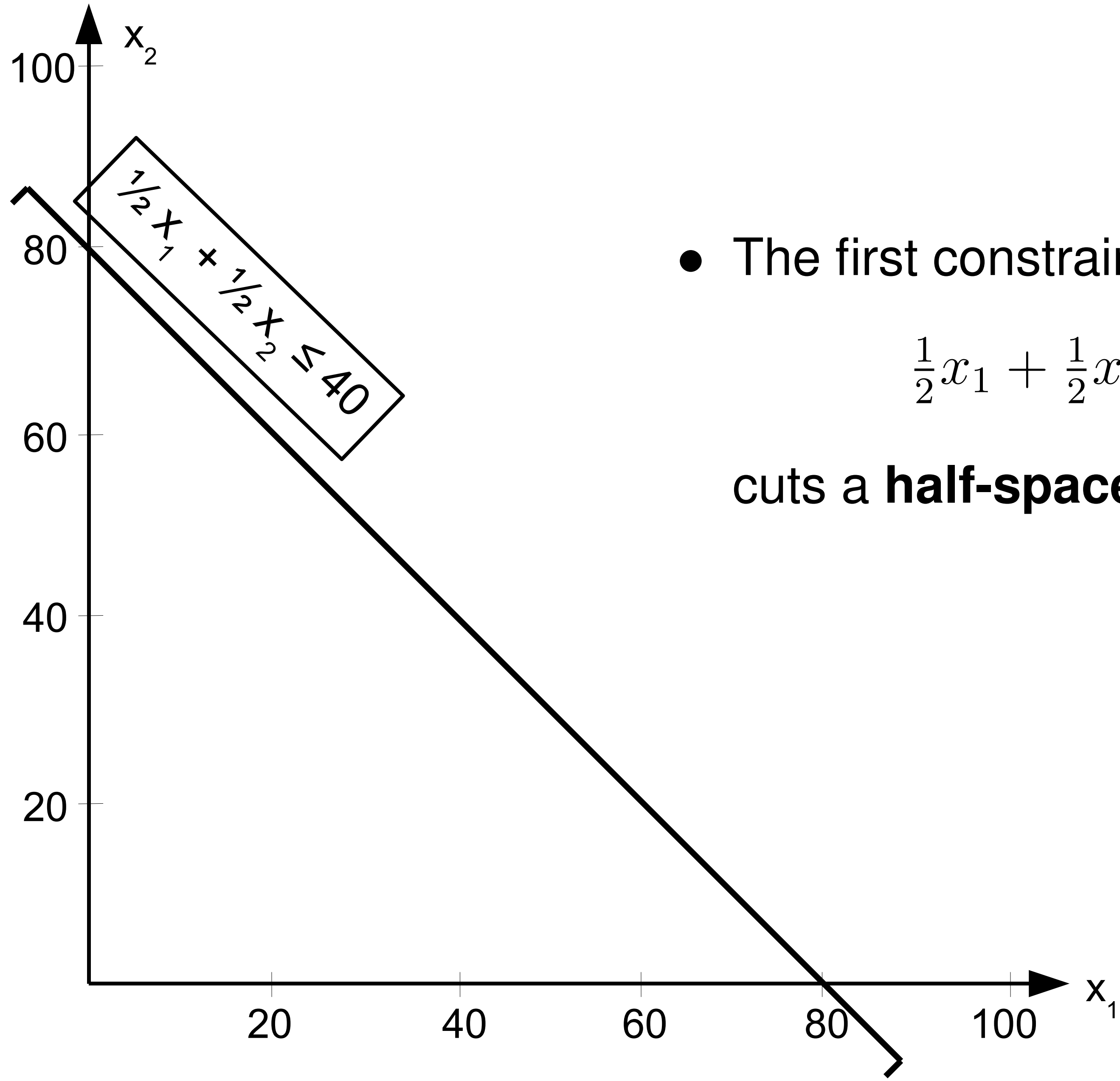
Graphical Solution



$$\begin{array}{ll} \max & 3x_1 + 4x_2 \\ \text{s.t.} & \frac{1}{2}x_1 + \frac{1}{2}x_2 \leq 40 \\ & x_1 + 2x_2 \leq 100 \\ & x_1, x_2 \geq 0 \end{array}$$

- 2 variables: 2 dimensions
- Nonnegative variables: feasible solutions lie in the positive orthant

Graphical Solution

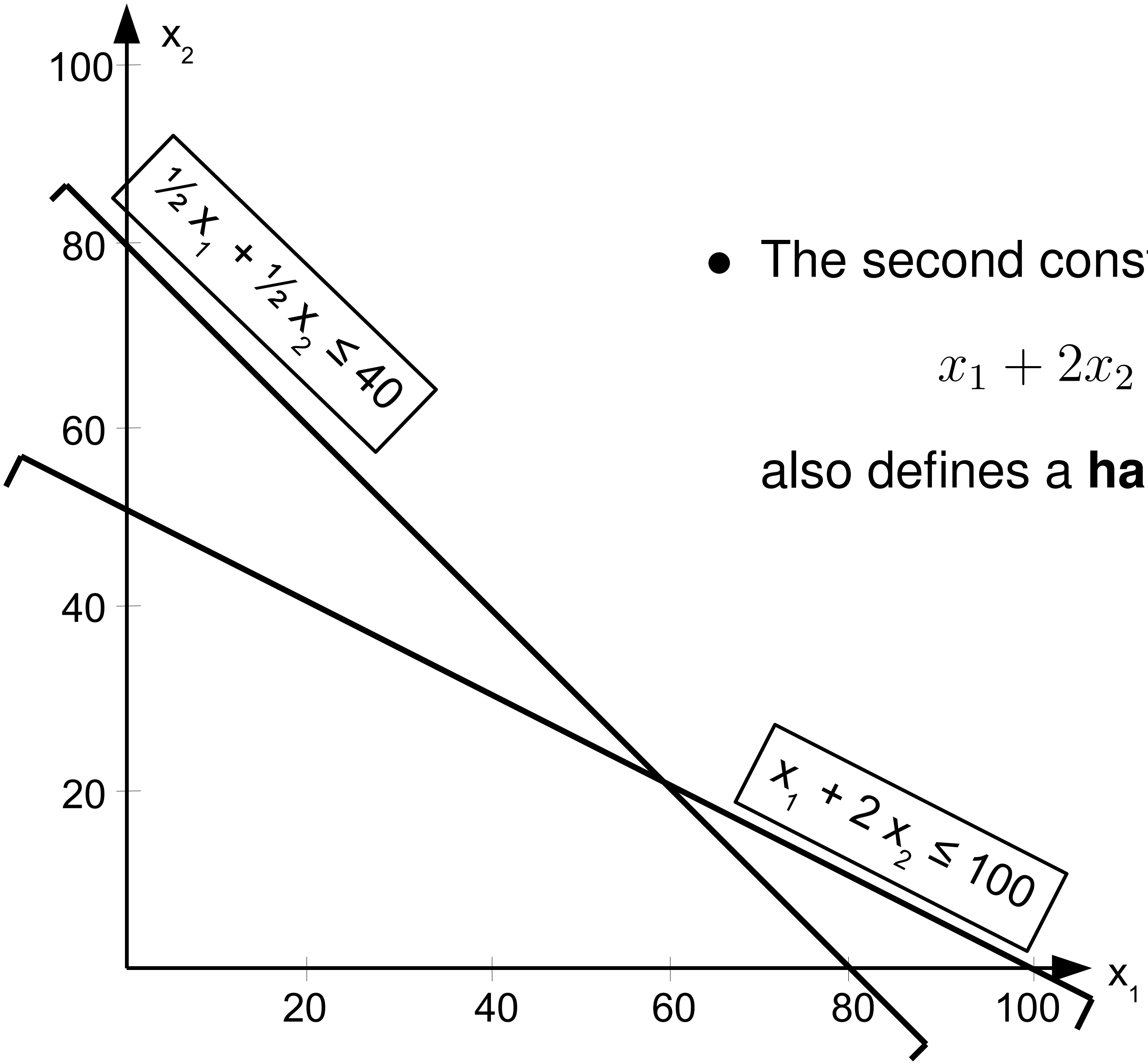


- The first constraint

$$\frac{1}{2}x_1 + \frac{1}{2}x_2 \leq 40$$

cuts a **half-space**

Graphical Solution

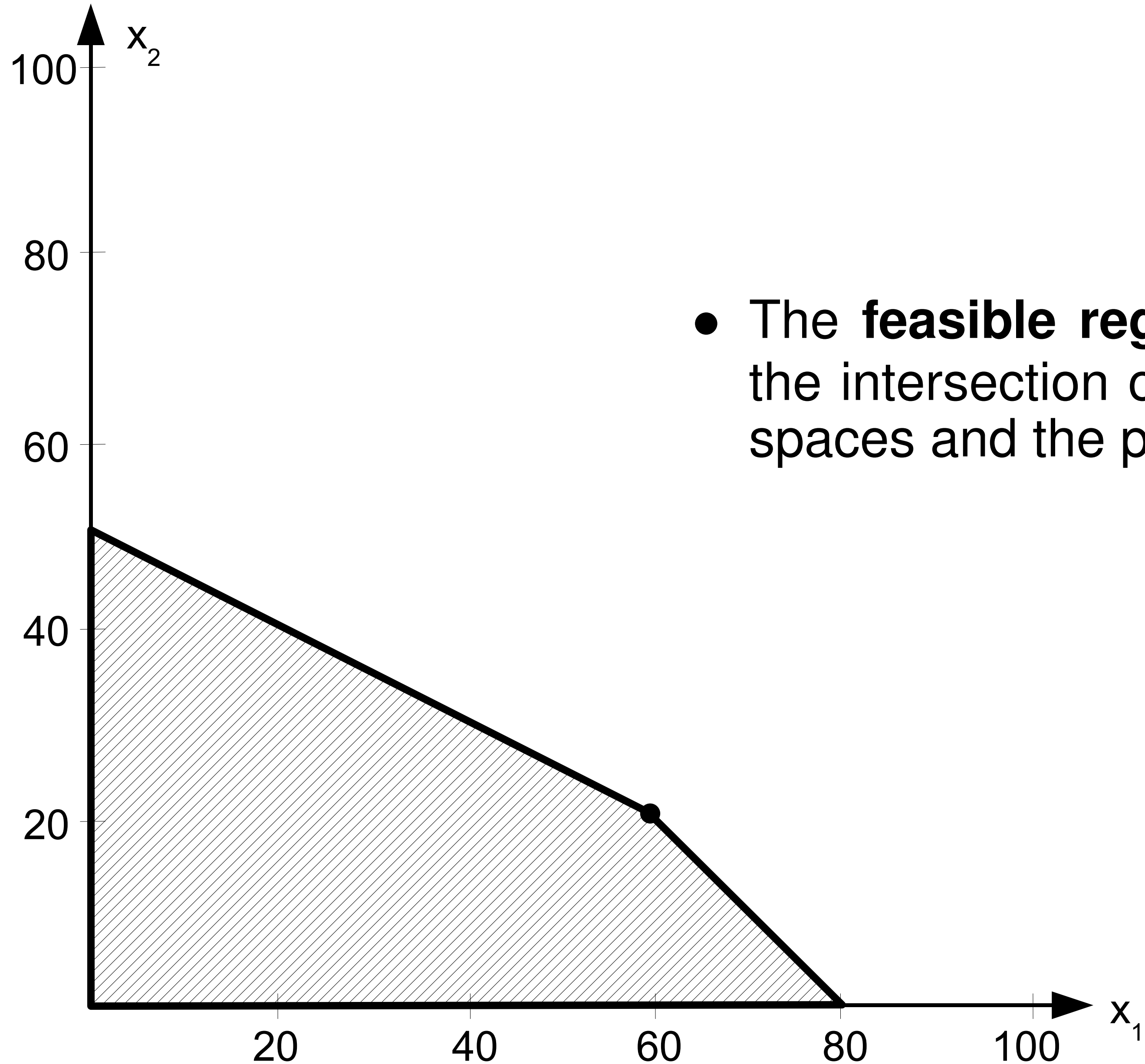


- The second constraint

$$x_1 + 2x_2 \leq 100$$

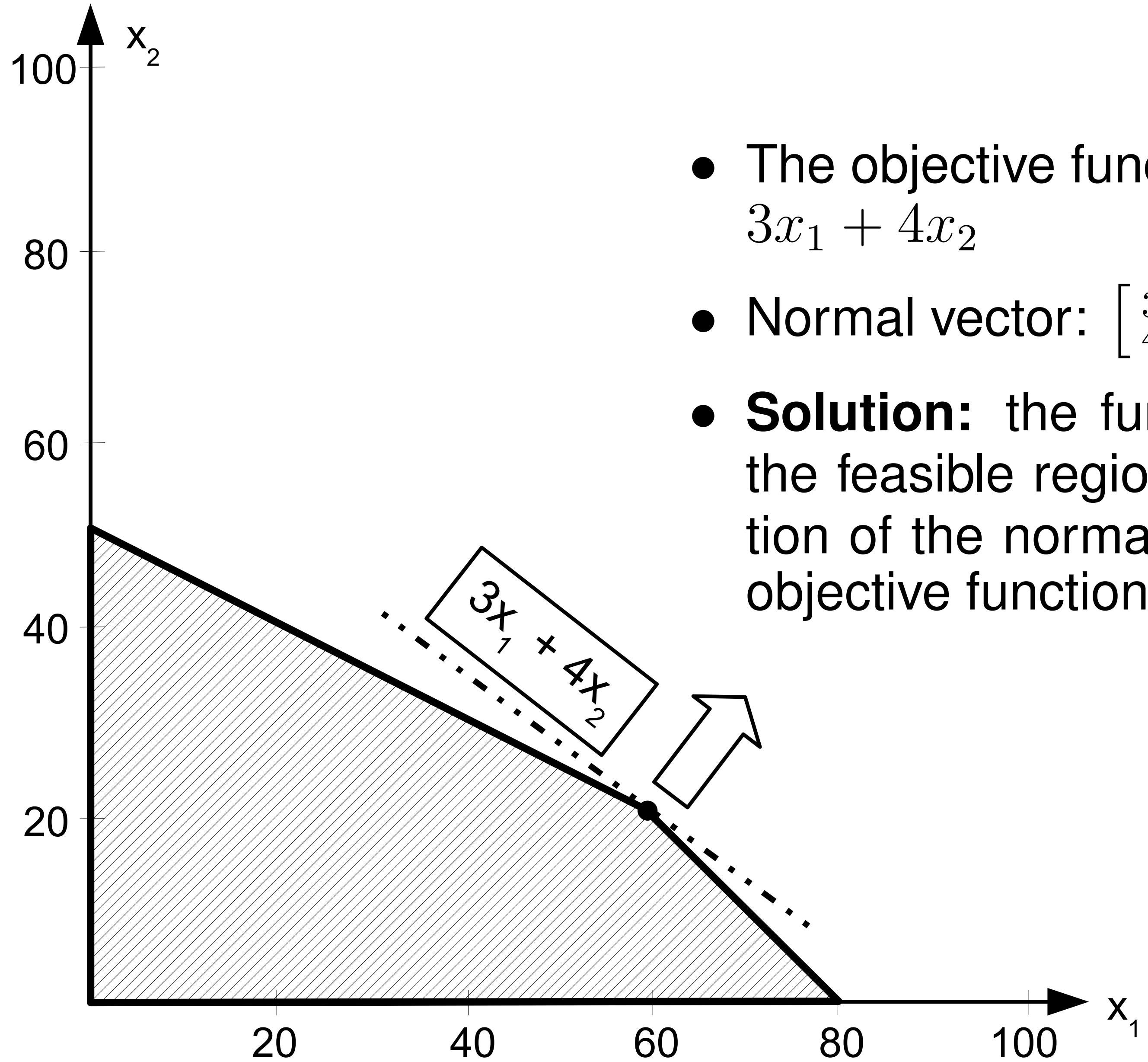
also defines a **half-space**

Graphical Solution



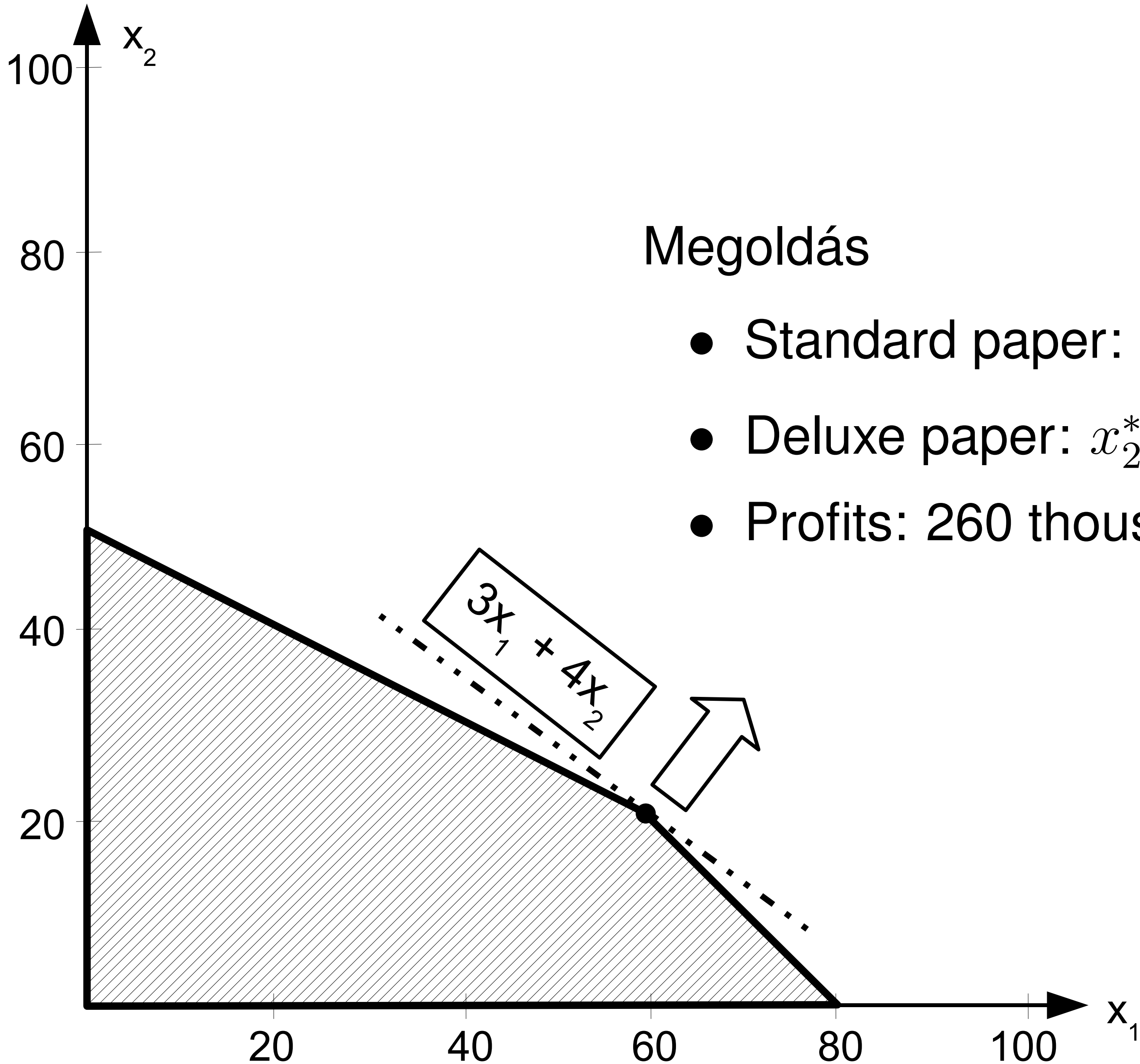
- The **feasible region** is exactly the intersection of the two half-spaces and the positive orthant

Graphical Solution



- The objective function: $3x_1 + 4x_2$
- Normal vector: $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$
- **Solution:** the furthest point of the feasible region in the direction of the normal vector of the objective function

Graphical Solution

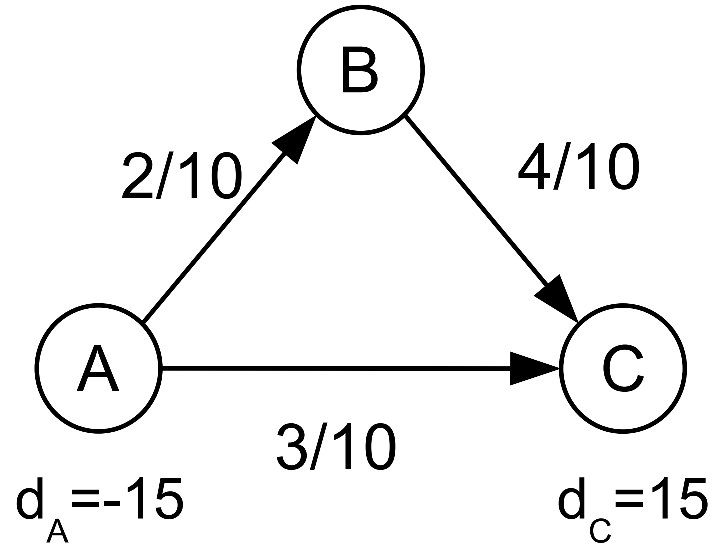


Megoldás

- Standard paper: $x_1^* = 60$ m^2
- Deluxe paper: $x_2^* = 20$ m^2
- Profits: 260 thousand USD

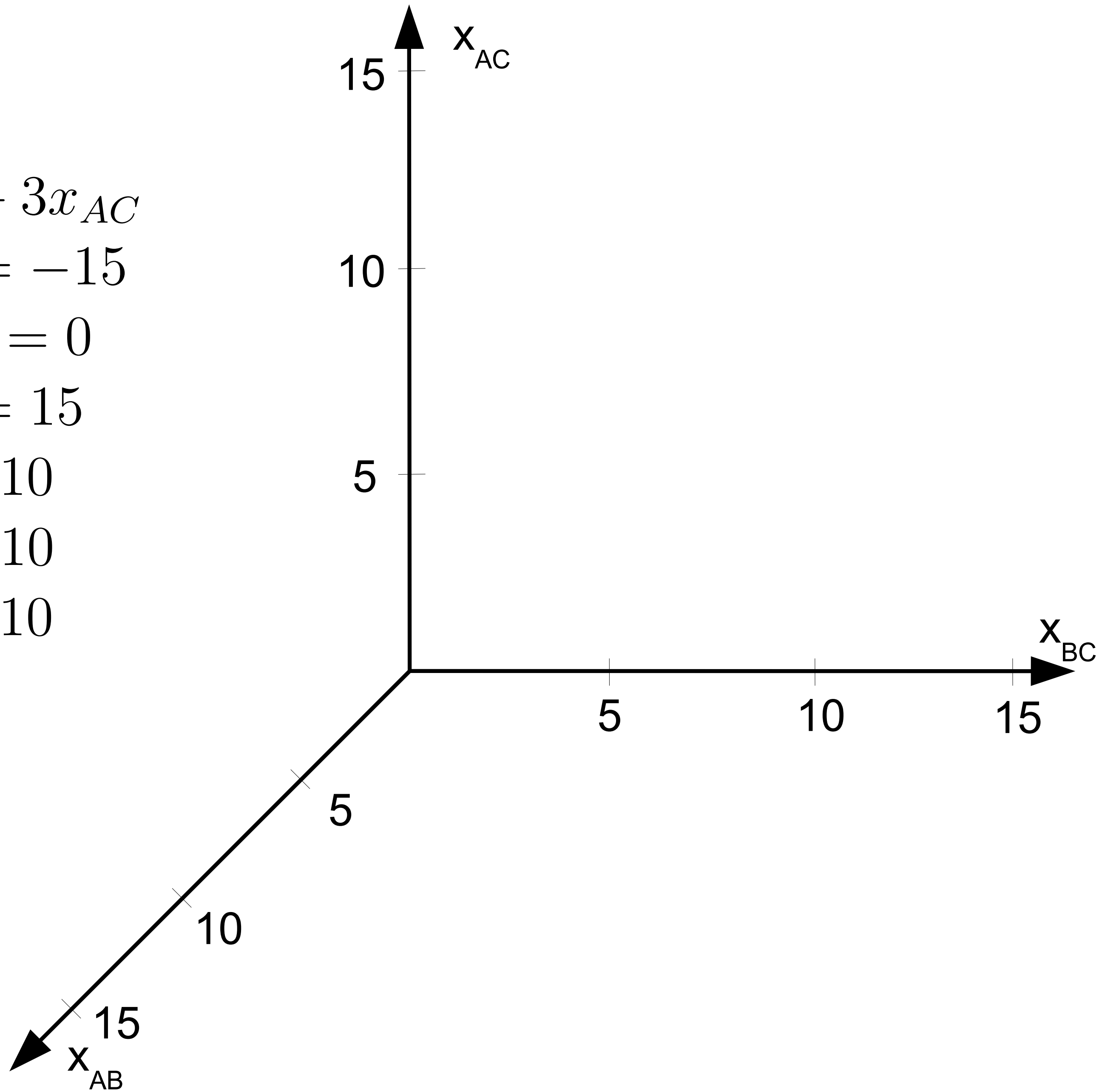
A Problem of Logistics

- Transshipment problem:
  - from point A to point C
  - 15 tonnes of goods
  - ship capacities are 10 tonnes each
  - cost ([thousand USD per tonne]): $c_{AB} = 2$, $c_{BC} = 4$, $c_{AC} = 3$
  - objective: minimize costs



A Problem of Logistics

$$
\begin{aligned}
\min \quad & 2x_{AB} + 4x_{BC} + 3x_{AC} \\
\text{s.t.} \quad & -x_{AB} - x_{AC} = -15 \\
& -x_{AB} + x_{BC} = 0 \\
& x_{BC} + x_{AC} = 15 \\
& 0 \leq x_{AB} \leq 10 \\
& 0 \leq x_{BC} \leq 10 \\
& 0 \leq x_{AC} \leq 10
\end{aligned}
$$

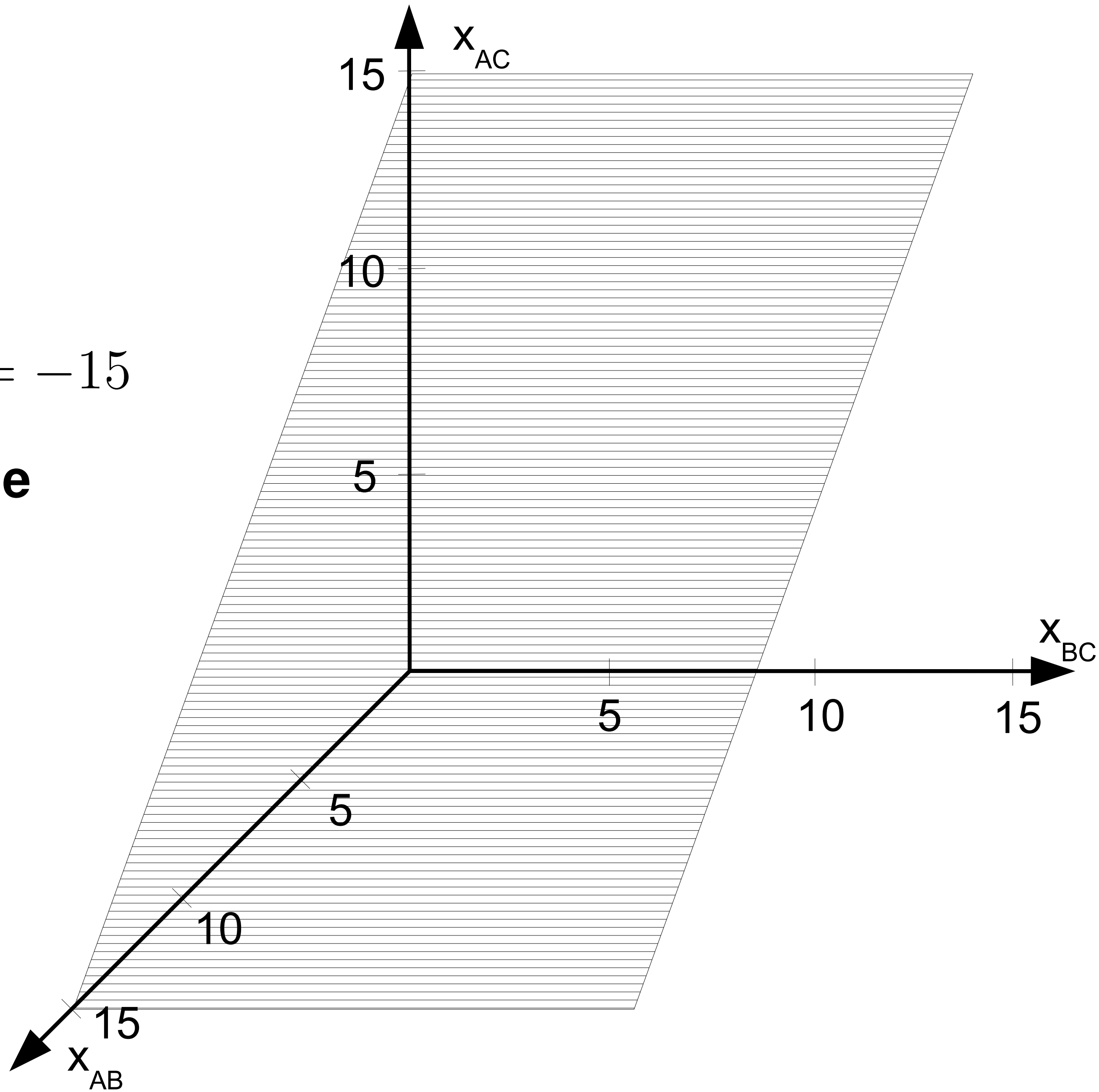


Graphical Solution

- The first constraint

$$-x_{AB} - x_{AC} = -15$$

  defines a **hyperplane**

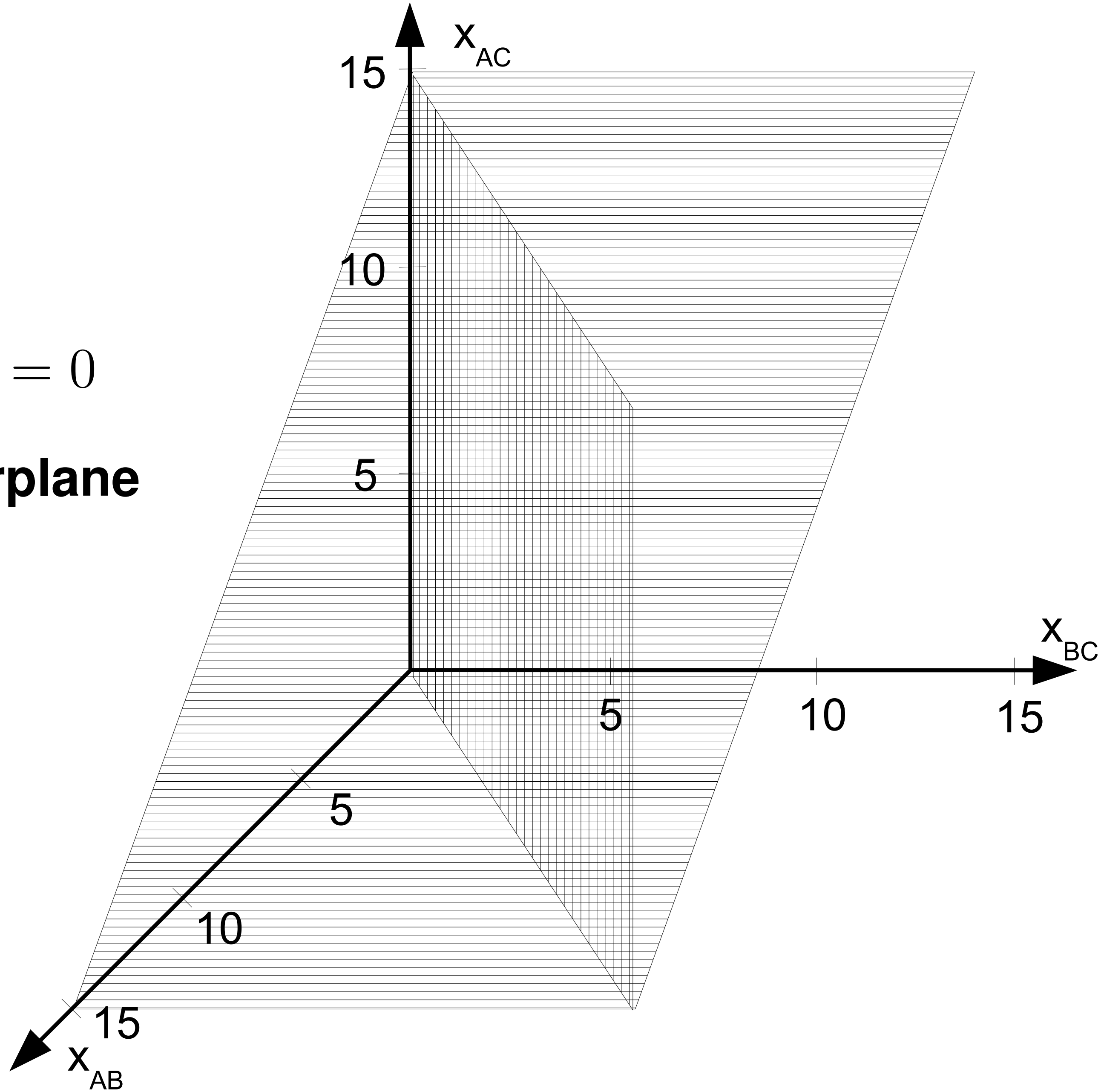


Graphical Solution

- The first constraint

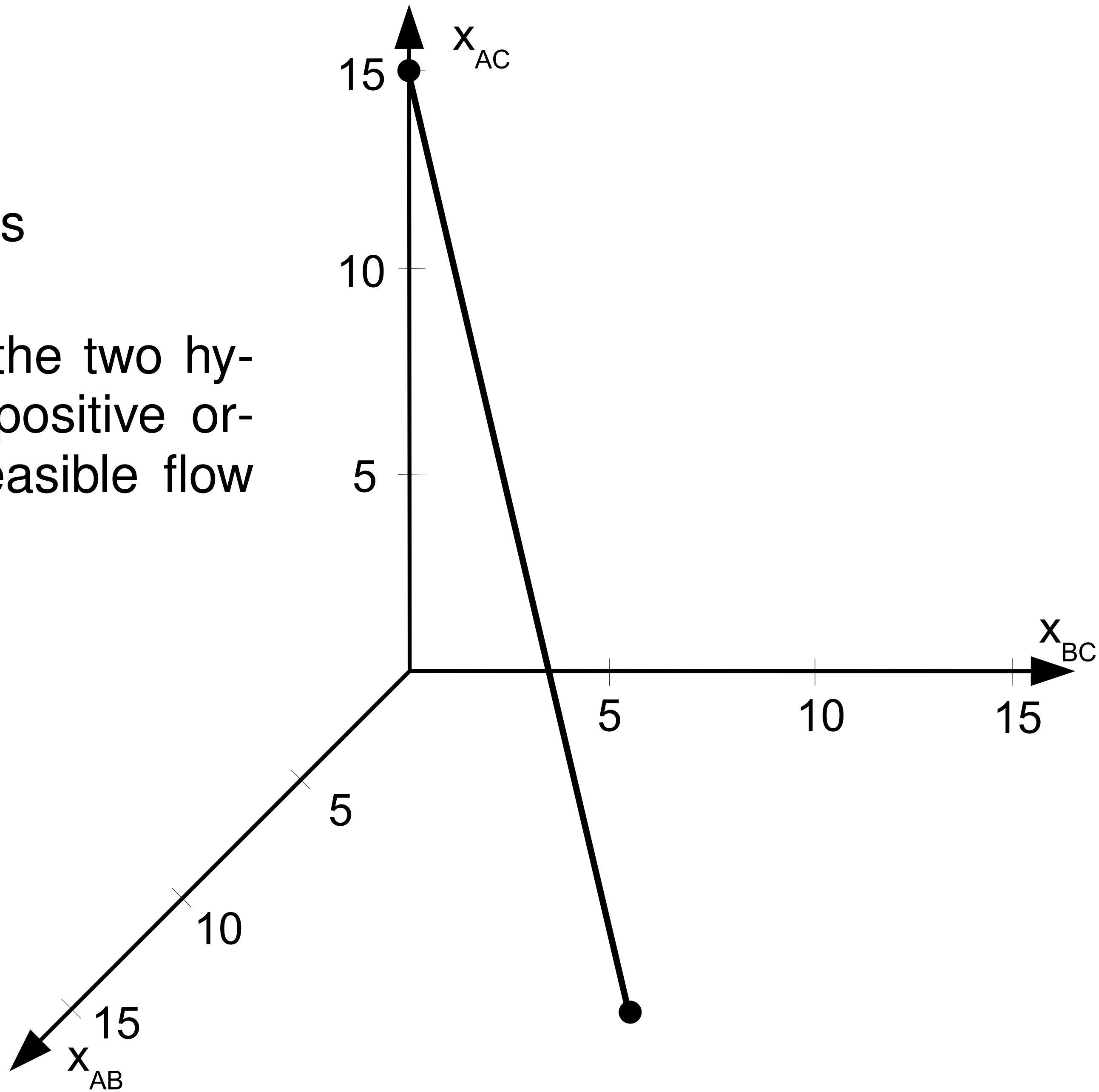
$$-x_{AB} + x_{BC} = 0$$

also defines a **hyperplane**



Graphical Solution

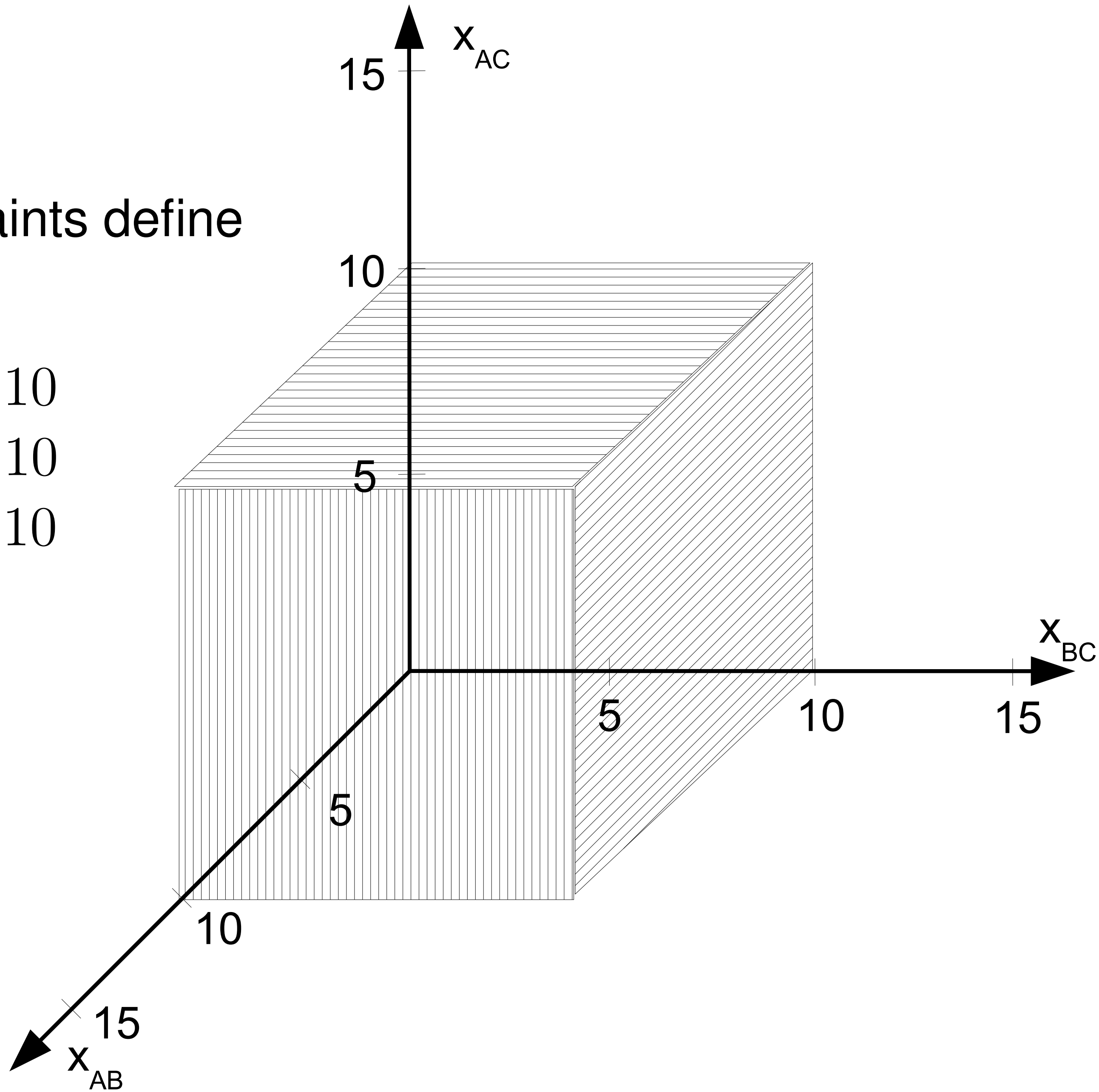
- The third constraint is redundant
- The intersection of the two hyperplanes and the positive orthant contains all feasible flow allocations



Graphical Solution

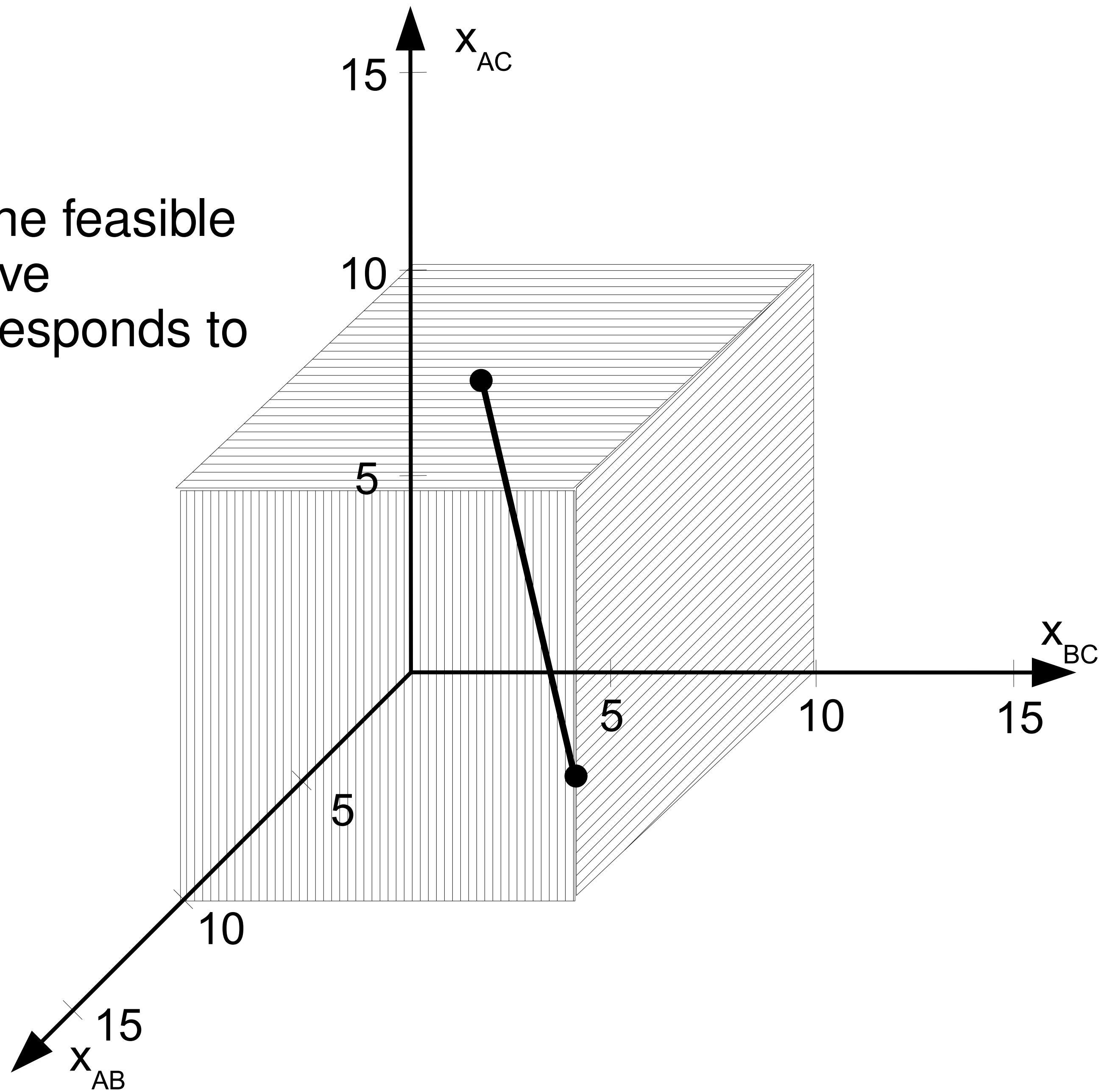
- The capacity constraints define a hyper-rectangle

$$0 \leq x_{AB} \leq 10$$
$$0 \leq x_{BC} \leq 10$$
$$0 \leq x_{AC} \leq 10$$



Graphical Solution

- The intersection of the feasible flow set and the above hyper-rectangle corresponds to the feasible region



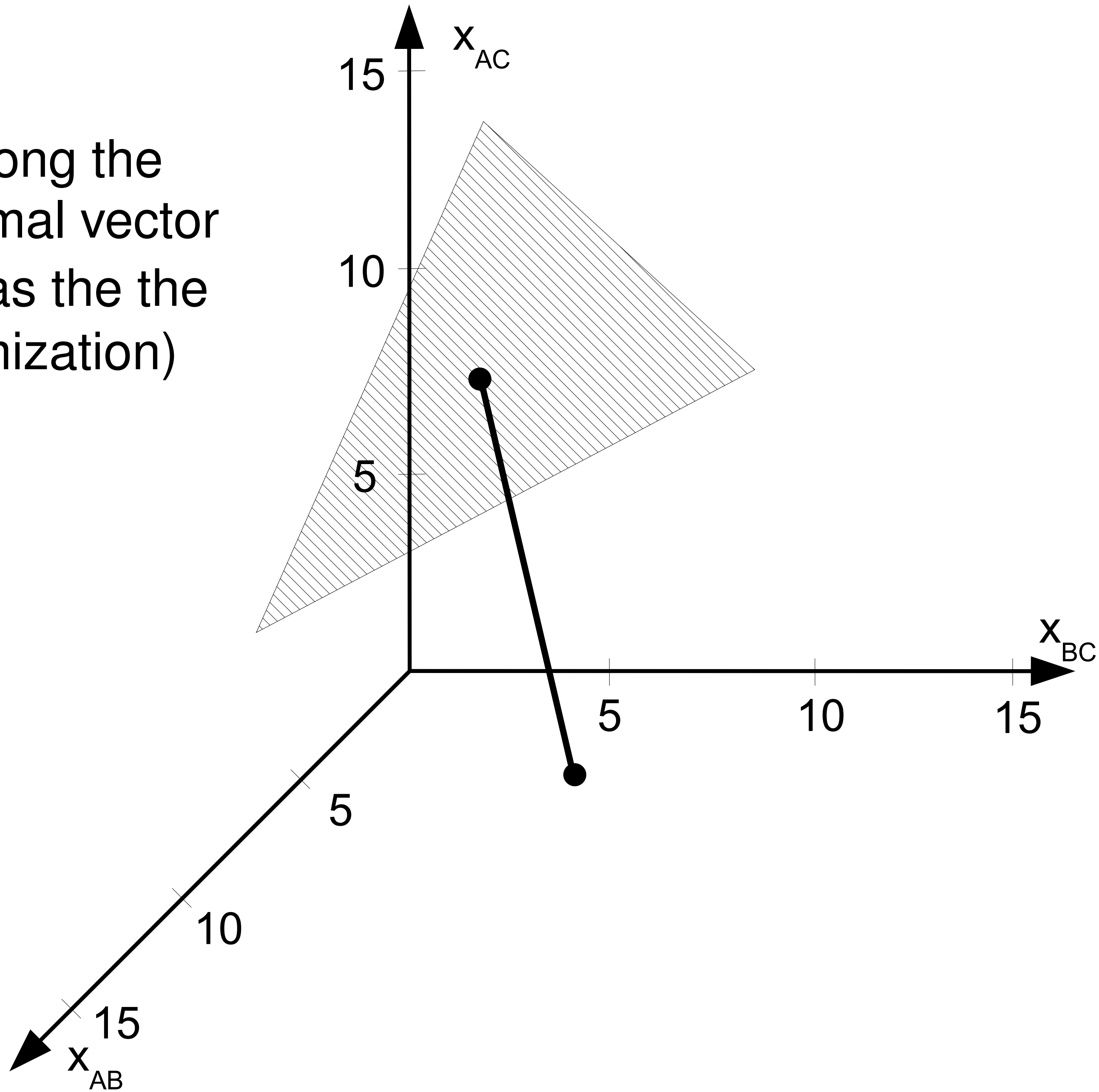
Graphical Solution

- The furthest point along the inverse(!) of the normal vector $[2 \quad 4 \quad 3]^T$ (inverse as the the optimization is minimization)

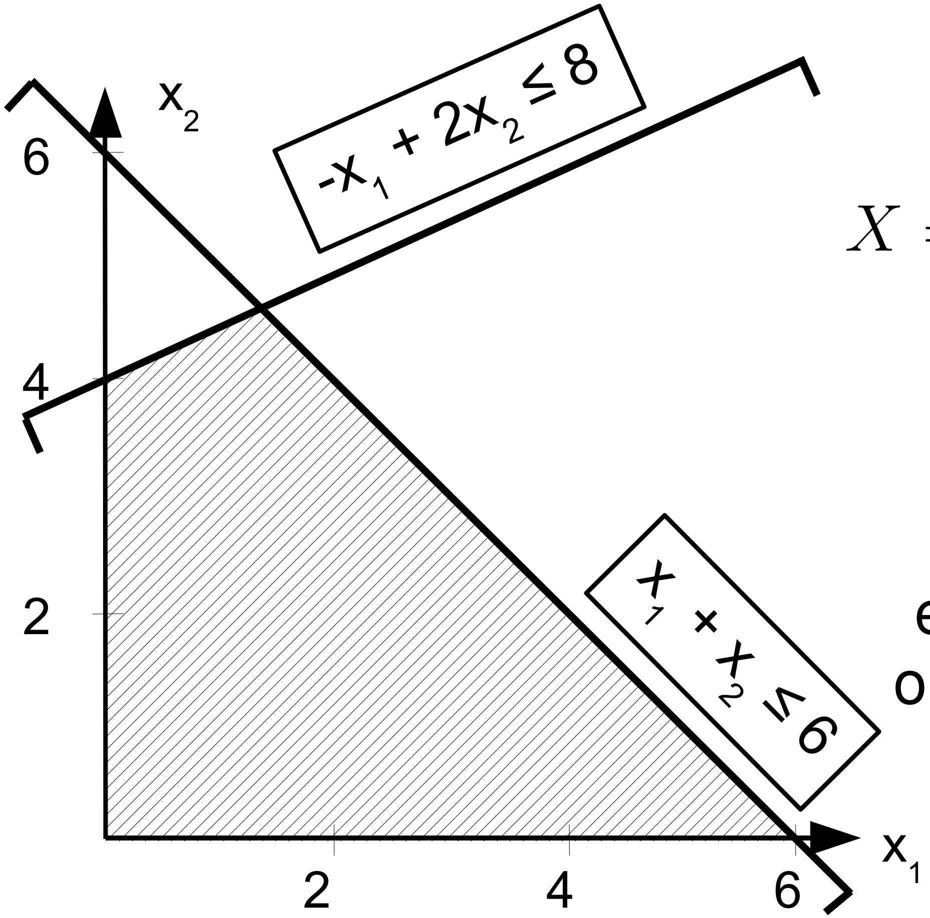
  $x_{AB} = 5$

  $x_{BC} = 5$

  $x_{AC} = 10$



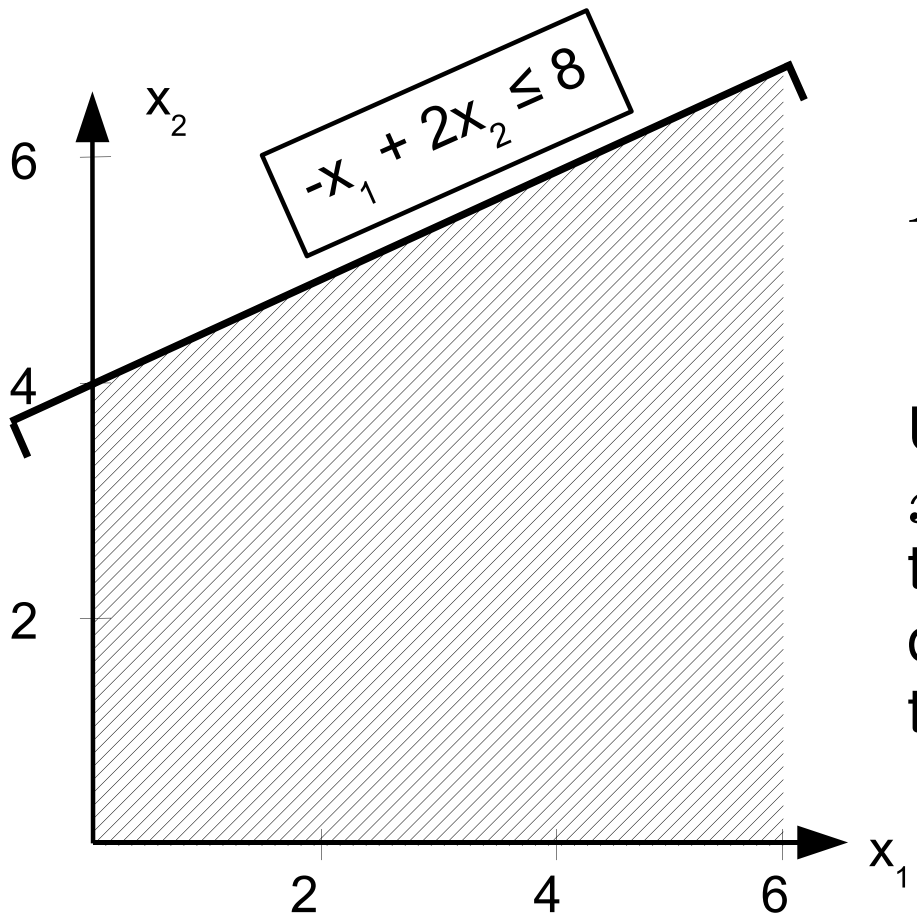
The Feasible Region



$$X = \{[x_1, x_2] : \quad \begin{aligned} x_1 + x_2 &\le 6 \\ -x_1 + 2x_2 &\le 8 \\ x_1, x_2 &\ge 0 \,\} \end{aligned}$$

Bounded: there is a large enough neighborhood of the origin that contains the entire feasible region

The Feasible Region

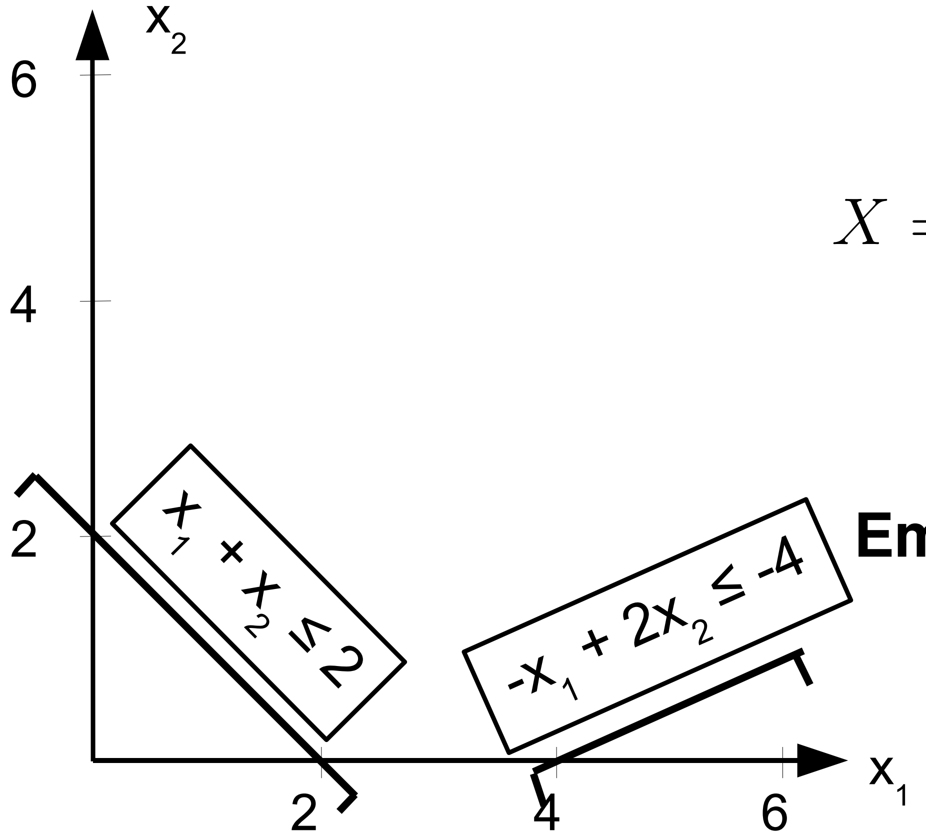


$$X = \{[x_1, x_2] : \quad -x_1 + 2x_2 \leq 8 \\ x_1, x_2 \geq 0 \}$$

Unbounded: there is point $\boldsymbol{x} \in X$ and vector $\boldsymbol{d}$ so that the **ray** from point $\boldsymbol{x}$ along direction $\boldsymbol{d}$ lies entirely within the feasible region:

$$\forall \lambda > 0 : \boldsymbol{x} + \lambda \boldsymbol{d} \in X$$

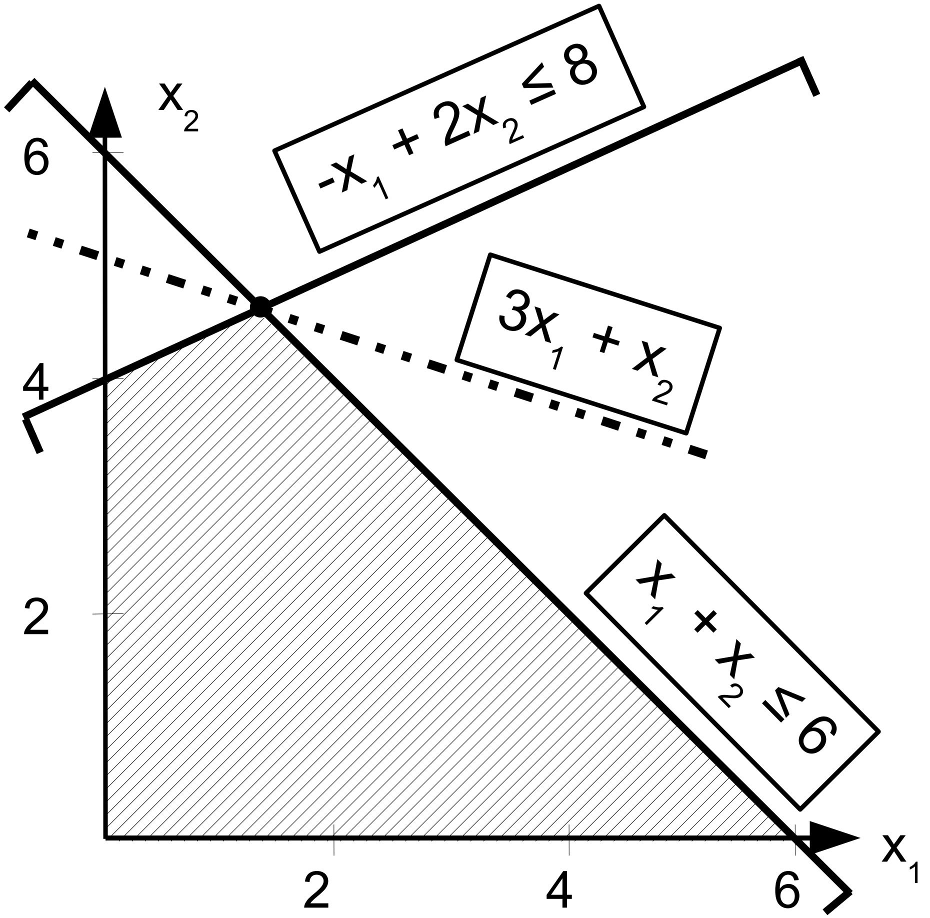
The Feasible Region



$$X = \{[x_1, x_2] : \quad x_1 + x_2 \leq 2$$
$$-x_1 + 2x_2 \leq -4$$
$$x_1, x_2 \geq 0 \;\}$$

Empty: the intersection of the half-spaces is empty

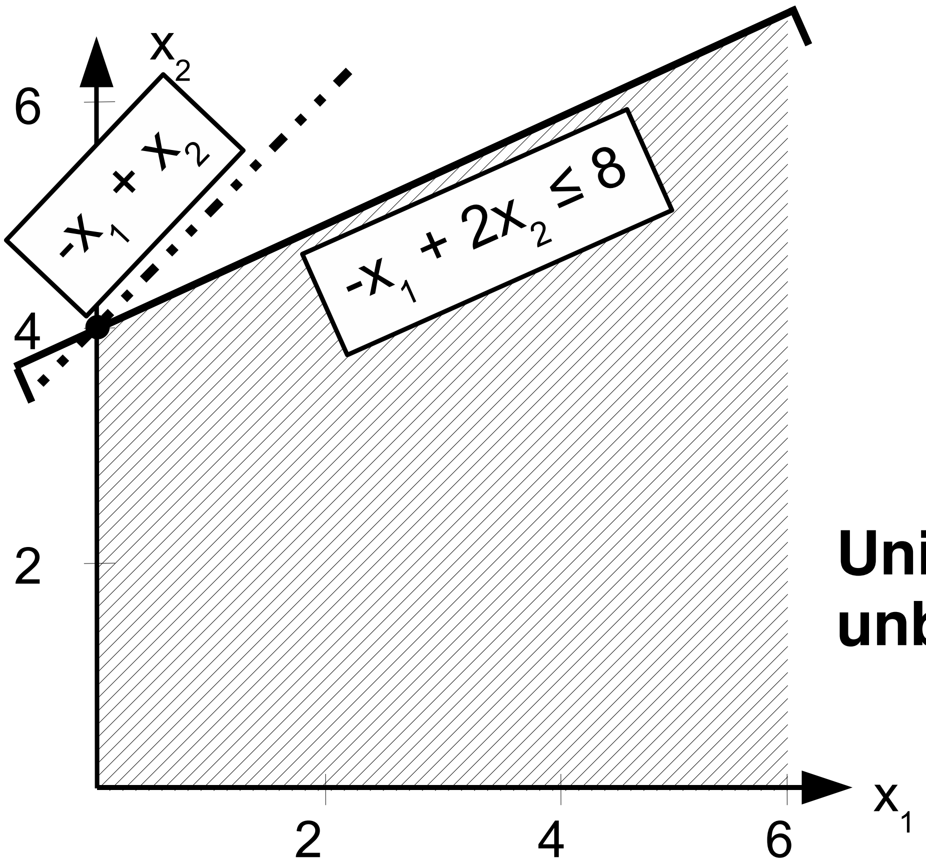
The Optimal Solution



$$
\begin{array}{rc}
\max & 3x_1 + x_2 \\
\text{s.t.} & x_1 + x_2 \leq 6 \\
 & -x_1 + 2x_2 \leq 8 \\
 & x_1, x_2 \geq 0
\end{array}
$$

Unique optimal solution on **bounded** feasible region: the objective is maximized at a unique point

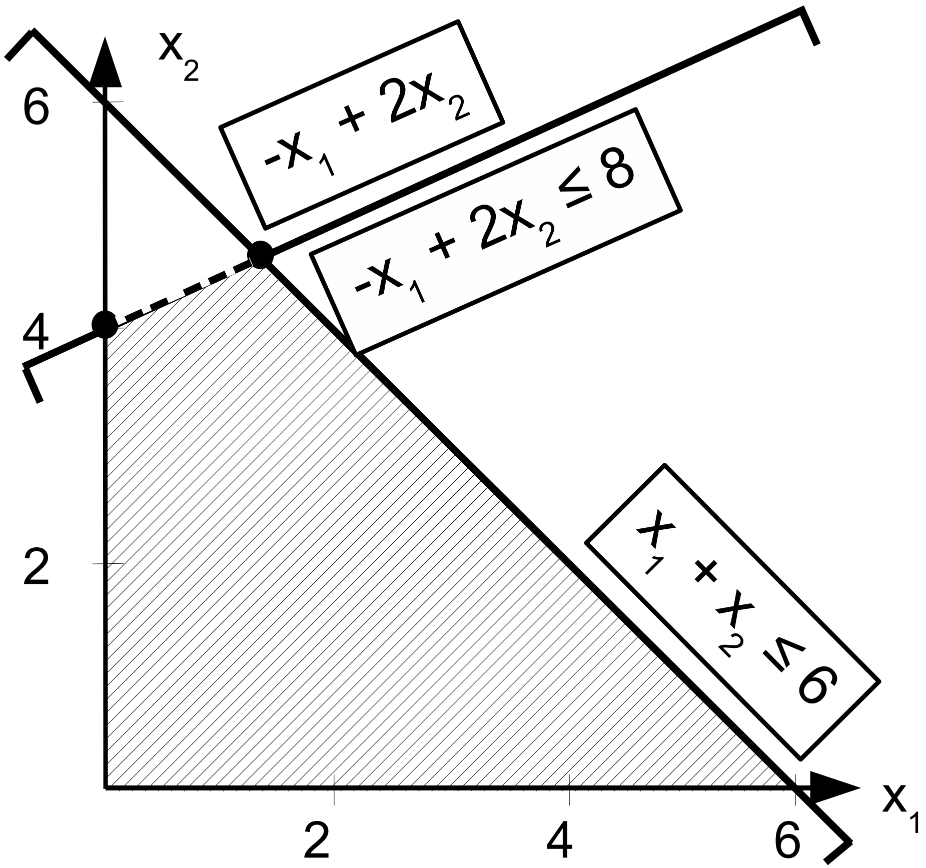
The Optimal Solution



$$\begin{array}{rl} \max & -x_1 + x_2 \\ \text{s.t.} & -x_1 + 2x_2 \leq 8 \\ & x_1, x_2 \geq 0 \end{array}$$

Unique optimal solution on an **unbounded** feasible region

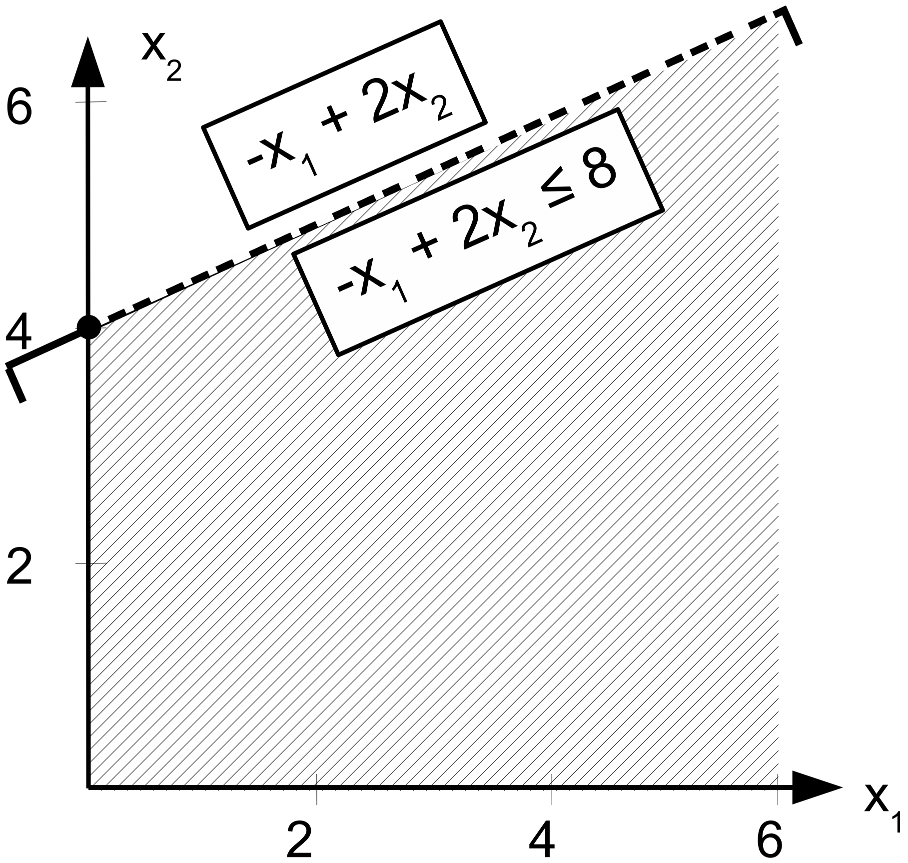
The Optimal Solution



$$
\begin{array}{rr}
\max & -x_1 + 2x_2 \\
\text{s.t.} & x_1 + x_2 \leq 6 \\
& -x_1 + 2x_2 \leq 8 \\
& x_1, x_2 \geq 0
\end{array}
$$

Alternative optimal solutions on a **bounded** feasible region: the objective function attains its maximum at more than one distinct points

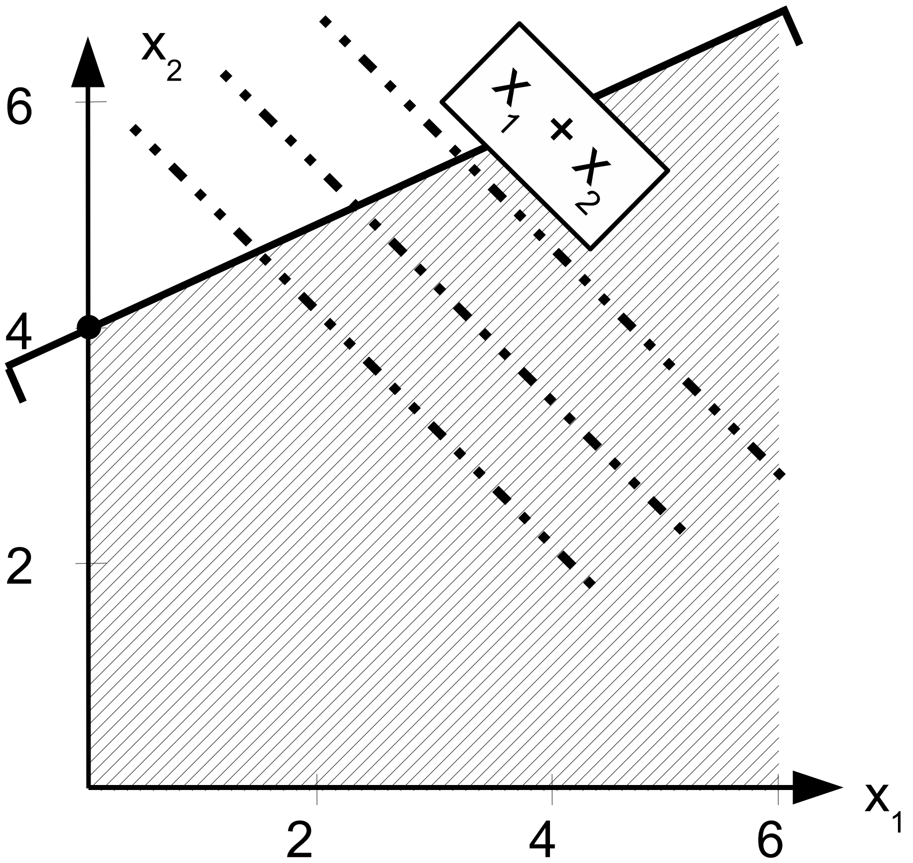
The Optimal Solution



$$
\begin{aligned}
\max \quad & -x_1 + 2x_2 \\
\text{s.t.} \quad & -x_1 + 2x_2 \le 8 \\
& x_1, x_2 \ge 0
\end{aligned}
$$

Alternative optimal solutions on an **unbounded** feasible region

The Optimal Solution



$$
\begin{array}{rr}
\max & x_1 + x_2 \\
\text{s.t.} & -x_1 + 2x_2 \leq 8 \\
 & x_1, x_2 \geq 0
\end{array}
$$

Unbounded optimal solution: the objective function value can be increased without limit inside the feasible region