Introduction to Linear Programming: A Summary

WARNING: this is just a summary of the material covered in the full slide-deck ***Introduction to Linear Programming*** *that will orient you as per the topics covered there; you are required to learn the full version, not just this summary!*

- Example: resource optimization
- Generic form of linear programs, basic definitions, matrix notation
- Miscellaneous topics: nonnegativity of variables, minimization and maximization, standard and canonical forms, transition between the two
- Notations and linear algebra

Optimal Resource Allocation

- **Exercise:** a paper mill manufactures two types of paper, standard and deluxe
  - $\frac{1}{2}$ m^3 of wood is needed to manufacture 1 m^2 of paper (both standard or deluxe)
  - producing 1 m^2 of standard paper takes 1 man-hour, whereas 1 m^2 of deluxe paper requires 2 man-hours
  - every week 40 m^3 wood and 100 man-hours of workforce is available
  - the profit is 3 thousand USD per 1 m^2 of standard paper and 4 thousand USD per 1 m^2 of deluxe paper
- **Question:** how much standard and how much deluxe paper should be produced by the paper mill per week to maximize profits?

Modeling 1: Selecting Variables

- **Optimal Resource Allocation/Product Mix problem:** optimal allocation of resources in order to maximize production profit
- Choose two variables:
  - x_1: the quantity produced from the standard paper [m^2]
  - x_2: the quantity produced from the deluxe paper [m^2]
- For instance, $x_1 = 12$, $x_2 = 20$ means: 12 m^2 of standard and 20 m^2 of deluxe paper produced, for which the mill uses
  - $\frac{1}{2} * 12 + \frac{1}{2} * 20 = 16$ m^3 wood and
  - $1 * 12 + 2 * 20 = 52$ man-hours of workforce,
  - meanwhile realizing $3 * 12 + 4 * 20 = 116$ thousands USD profits

Modeling 2: Constraints

- **Resource constraint:** the available quantity of wood (40 m^3) limits the amount of paper that can be produced:

$$\frac{1}{2}x_1 + \frac{1}{2}x_2 \leq 40$$

- **Labor constraint:** the available workforce (100 man-hours) also limits the possible production mixes:

$$x_1 + 2x_2 \leq 100$$

- **Nonnegativity:**

$$x_1 \geq 0, \quad x_2 \geq 0$$

Modeling 3: The Objective Function

- The profits: $3x_1 + 4x_2$ [thousand USD]
- **Objective:** to maximize profits:

$$\max 3x_1 + 4x_2$$

in a way so that the amount of wood and workforce used does not exceed the available quantities

Linear Program

$$
\begin{array}{lrcrcr}
\max & 3x_1 & + & 4x_2 & & \\
\\
\text{s.t.} & \frac{1}{2}x_1 & + & \frac{1}{2}x_2 & \leq & 40 \\
 & x_1 & + & 2x_2 & \leq & 100 \\
\\
 & x_1, & & x_2 & \geq & 0
\end{array}
$$

Linear programs: Basic Definitions

- Maximization of the **objective function** that is a linear function of the **decision variables/activities**, or the minimization of a linear **cost function**
- The solution meets the **constraints** that are also linear functions of the variables
- The linear scaling constants are called **objective function coefficients** and **constraint coefficients**
- The combinations of variables x_1, x_2 that meet the constraints are called **feasible solutions** or **feasible points**
- The set of feasible solutions is called the **feasible region**
- The feasible solutions that maximize the objective function (minimize the cost function) are called the **optimal (feasible) solutions** (there can be more than one)
- Decision variables may be subject to nonnegativity or nonpositivity constraints

The General Form of Linear Programs

$$
\begin{array}{lccccccccc}
\max & c_1x_1 & + & c_2x_2 & + & \ldots & + & c_nx_n & & \\
\text{s.t.} & a_{11}x_1 & + & a_{12}x_2 & + & \ldots & + & a_{1n}x_n & \leq & b_1 \\
 & a_{21}x_1 & + & a_{22}x_2 & + & \ldots & + & a_{2n}x_n & \leq & b_2 \\
 & \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\
 & a_{m1}x_1 & + & a_{m2}x_2 & + & \ldots & + & a_{mn}x_n & \leq & b_m \\
 & & & & & & & & & \\
 & x_1, & & x_2, & & \ldots, & & x_n & \geq & 0
\end{array}
$$

The General Form of Linear Programs

- m: the number of **rows**, i.e., the number of constraints
- n: the number of **columns**, i.e., the number of variables
- c_j: the objective coefficient for the j-th variable
- $\sum_{j=1}^{n} c_j x_j$: the objective/cost function
- $\sum_{j=1}^{n} a_{ij} x_j \leq b_i$: the i-th constraint
  - a_{ij}: constraint coefficients
  - b_i: the i-th "right-hand-side" (RHS)

The Matrix Form of Linear Programs

$$
\begin{array}{lccc}
\max & \begin{bmatrix} c_1 & c_2 & \dots & c_n \end{bmatrix} \boldsymbol{x} & & \\
\text{s.t.} & \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \boldsymbol{x} & \leq & \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \\
 & \boldsymbol{x} & \geq & \mathbf{0}
\end{array}
$$

$$
\begin{array}{rc}
\max & \boldsymbol{c}^T \boldsymbol{x} \\
\text{s.t.} & \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \\
 & \boldsymbol{x} \geq \mathbf{0}
\end{array}
$$

Direction of Optimization

- If the objective function represents
  - profits: **maximization**
  - cost: **minimization**
- Conversion:

$$\max\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\} = \\ -1 * \min\{-\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$$

Constraints: Forms

- **Inequality and equality:** there are two types of constrains in the transportation problem
  - Supply constraint: $\sum_{j=1}^{4} x_{ij} \leq \text{capacity}_i$
  - Demand constraint: $\sum_{i=1}^{3} x_{ij} = \text{demand}_j$
- Conversion: inequality $\rightarrow$ equality
  - "$\leq$" type inequality: by **adding** an artificial **slack** variable

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i \Longleftrightarrow \sum_{j=1}^{n} a_{ij}x_j + x_{s_i} = b_i, \quad x_{s_i} \geq 0$$

Constraints: Forms

- Conversion: inequality $\rightarrow$ equality
  - "$\geq$" type inequality: by **subtracting** an artificial **slack** variable

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i \Longleftrightarrow \sum_{j=1}^{n} a_{ij}x_j - x_{s_i} = b_i, \quad x_{s_i} \geq 0$$

- Conversion: equality $\rightarrow$ inequality
  - a "$=$" type constraint can be substituted with a "$\leq$" type and a "$\geq$" type constraint

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \Longleftrightarrow \sum_{j=1}^{n} a_{ij}x_j \leq b_i$$

$$\sum_{j=1}^{n} a_{ij}x_j \geq b_i$$

Nonnegativity

- In practice the variables are almost always constrained as nonnegative
- Substituting a nonpositive variable: $x_j = -x'_j$

$$\begin{array}{ccc} x_j \leq 0 & & x'_j \geq 0 \\ a_{ij}x_j & \Longleftrightarrow & -a_{ij}x'_j \\ c_jx_j & & -c_jx'_j \end{array}$$

- Substituting a free variable: $x_j = x'_j - x''_j$

$$\begin{array}{ccc} x_j \overset{\leq}{\underset{\geq}{=}} 0 & & x'_j \geq 0,\ x''_j \geq 0 \\ a_{ij}x_j & \Longleftrightarrow & a_{ij}(x'_j - x''_j) \\ c_jx_j & & c_j(x'_j - x''_j) \end{array}$$

The Canonical and the Standard Forms

	Minimization	Maximization
Standard form	$\min \boldsymbol{c}^T\boldsymbol{x}$ s.t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ $\boldsymbol{x} \geq \boldsymbol{0}$	$\max \boldsymbol{c}^T\boldsymbol{x}$ s.t. $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ $\boldsymbol{x} \geq \boldsymbol{0}$
Canonical form	$\min \boldsymbol{c}^T\boldsymbol{x}$ s.t. $\boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}$ $\boldsymbol{x} \geq \boldsymbol{0}$	$\max \boldsymbol{c}^T\boldsymbol{x}$ s.t. $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ $\boldsymbol{x} \geq \boldsymbol{0}$

Linear Algebra: A Reminder

- Vectors (column and row), operations on vectors, linear independence and the basis
- Matrices, operations on matrices, matrix rank and singularity, the determinant
- Solving systems of linear equations
- Make sure you understand all the above: we will use these basic linear algebraic definitions everywhere and silently assume that you are familiar with the background