On a problem of Rényi and Katona

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Abstract: We are dealing with the classical problem of determining the minimum size of a separating system consisting of sets of size $k$. The problem was raised by Rényi, the first and most important results are due to Katona; Wegener, Luzgin and Ahlswede also proved important bounds. We give a simple, short proof of a strengthening of Katona’s main theorem determining the minimum size of a separating system of $k$-sets.

Keywords: Separating system, $k$-set, search

1 Introduction and results

A set system is said to be a separating system if any two elements of the underlying set can be separated by some set of the system. More formally:

Definition 1 Let $H$ be a finite set. The system $A \subseteq 2^H$ is a separating system if for any $x, y \in H$, $x \neq y$ : $\exists S \in A$, such that $x \in S$, $y \notin S$ or $x \notin S$, $y \in S$.

Separating systems were introduced by Alfréd Rényi [6] in 1961 concerning information-theoretic problems. The problem of finding the minimum size of a separating system containing sets of size $k$ was also raised by Rényi.

Definition 2 Let $m$ and $k$ be positive integers, such that $k < \frac{m}{2}$. Let us denote the smallest size of a separating system $A \subseteq 2^{[m]}$ of sets of size exactly $k$, size at most $k$, and average size at most $k$, by $n(m,k)$, $n'(m,k)$, and $n^*(m,k)$, respectively.

It is obvious that

Claim 3 $n^*(m,k) \leq n'(m,k) \leq n(m,k)$.

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Rényi’s problem was to determine the number \( n(m,k) \). In 1966 Katona, using the main theorem in [2] showed

**Theorem 4 (Katona)** For \( k < \frac{m}{2} \), \( n'(m,k) = n(m,k) \).

In 2008 Ahlswede showed [1, Appendix]

**Theorem 5 (Ahlswede)** For \( k < \frac{m}{2} \), \( n^*(m,k) = n(m,k) \).

We give a short, simple proof of both theorems in Section 2. Katona’s main theorem in [2] is the following.

**Theorem 6 (Katona)** For \( k < \frac{m}{2} \), \( n(m,k) \) is equal to the least number \( n \), for which there exists a system of non-negative integers \( s_0, s_1, \ldots, s_n \) satisfying the following three conditions:

\[
\sum_{i=0}^{n} i \cdot s_i = kn,
\]

\[
\sum_{i=0}^{n} s_i = m,
\]

\[
s_i \leq \binom{n}{i} \quad i = 0, 1, \ldots, n.
\]

We prove the following strengthening of this theorem.

**Theorem 7** For \( k < \frac{m}{2} \), \( n(m,k) \) is equal to the least number \( n \), for which there exist natural numbers \( j \leq n - 1 \) and \( a < \binom{n}{j+1} \), such that

\[
\sum_{i=0}^{j} i \cdot \binom{n}{i} + a(j+1) \leq kn,
\]

\[
\sum_{i=0}^{j} \binom{n}{i} + a = m.
\]

Katona mentions [2] that though Theorem 6 determines \( n(m,k) \) implicitly, it cannot be used to compute the value of \( n(m,k) \). On the other hand, using Theorem 7 it is easy to compute \( n(m,k) \): first fix \( n \) and \( k \) and find the maximum \( m \) satisfying (4) and (5). Let this maximum be \( M(n,k) \). Condition (4) is equivalent to

\[
\sum_{i=0}^{j-1} \binom{n-1}{i} + a(j+1) \frac{1}{n} \leq k,
\]

where \( \frac{a(j+1)}{n} < \binom{n-1}{j} \), thus the maximum possible values for \( j \) and \( a \) are easy to find. Therefore, by (5) we have \( M(n,k) \). Now \( n(m,k) \) is the smallest \( n \), for which \( m \leq M(n,k) \). In Section 2 we will also see that not just the size of a minimum separating system of \( k \)-sets is easy to determine but it is also easy to give such a system.

It is worth mentioning that a closed formula for \( n(m,k) \) is not known. The best known lower bound (based on a nice entropy approach) is due to Katona [2], while the best known upper bound is due to Wegener [7] and Luzgin [4]. In 2002 Katona showed [3] that Theorem 6 can be used to obtain really good approximate solutions, while in 2008 Ahlswede proved [1] that the entropy type bound of Katona is asymptotically tight.
2 Proofs

Let \( H \subseteq 2^{[m]} \) be a set system of size \( n \) and consider any linear order of its sets. The incidence matrix of \( H \) is the 0–1 matrix \( M(H) = (m_{ij})_{n,m} \), where \( m_{ij} = 1 \) if the \( i \)th set of \( H \) contains the element \( j \) and 0 otherwise. Henceforth, all matrices in this paper are binary. A matrix will be called simple, if it does not contain identical columns. The weight of a row or a column \( A \) is defined as the number of 1’s in \( A \) and is denoted by \( w(A) \). We use the following two notions of Katona [2]: a matrix is called admissible if the weights of any two rows are the same, and a matrix is called quasi-admissible if the weights of any two rows differ by at most one.

It is easy to see [5] that a set system \( H \) is separating if and only if \( M(H) \) is a simple matrix and therefore \( n(m, k) \) (\( n'(m, k) \)) is the smallest number \( n \), such that an \( n \times m \) simple matrix with row weights exactly (at most) \( k \) exists.

First we give a short proof of Theorem 4.

**Proof of Theorem 4:** By Claim 3 we only have to prove \( n(m, k) \leq n'(m, k) \). For this, it suffices to show that if there exists an \( n \times m \) simple matrix \( M \) with row weights at most \( k \), then there exists an \( n \times m \) simple matrix \( M' \), where every row has weight \( k \). Let \( M \) be an \( n \times m \) simple matrix \( M \) with row weights at most \( k \), such that the number of 1’s in \( M \) is maximum. We show that every row of \( M \) has weight \( k \). Assume to the contrary that a row \( A \) of \( M \) exists, such that \( w(A) < k \). For the sake of convenience let us assume that \( A \) is the first row of \( M \). Since \( w(A) < k < \frac{m}{2} \), the number of 0’s is greater than the number of 1’s in \( A \). Therefore, there exists a column \( C \) of \( M \), such that the first entry of \( C \) is 0 and \( M \) does not contain the column which differs from \( C \) only in the first entry. Thus if we change the first entry of column \( C \) to 1, we obtain a simple matrix \( M' \) with row weights at most \( k \), such that \( M' \) contains more 1’s than \( M \), a contradiction. \( \square \)

Let \( r(m, k) \) be the least number \( n \), for which there exist numbers \( j \) and \( a \), such that \( j \leq n - 1 \), \( 0 \leq a < \binom{n}{j+1} \) and equations (4) and (5) hold.

**Lemma 8** \( n^*(m, k) = r(m, k) \).

**Proof:** First we show that \( n^*(m, k) \leq r(m, k) \). Let \( n = r(m, k) \), and \( j \), \( a \) the numbers for which (4) and (5) hold. Let us consider a matrix \( M \) consisting of every column of length \( n \) and weight at most \( j \) and a different columns of length \( n \) and weight \( j + 1 \). \( M \) is obviously simple and contains \( n \) rows, furthermore by (4) and (5) \( M \) contains \( m \) columns and at most \( kn \) 1’s. The existence of such a matrix proves the inequality. In order to prove \( r(m, k) \leq n^*(m, k) \) let \( n = n^*(m, k) \) and let \( M \) be a simple \( n \times m \) matrix containing at most \( kn \) 1’s, such that the number of 1’s is minimum. We show that for some \( j < n \) every column of \( M \) has weight at most \( j + 1 \) and every column of weight at most \( j \) appears in \( M \). Now if we let \( a \) be the number of columns of weight \( j + 1 \) then it is easy to check that for \( j \) and \( a \) the equations (4) and (5) hold, from which the inequality follows. For this, we have to show that if a column \( A \) of length \( n \) appears in \( M \), then every column \( B \) of length \( n \) and weight less than \( w(A) \) also appears in \( M \). This is easy to see: if \( w(B) < w(A) \) and \( B \) is not a column of \( M \), then by deleting \( A \) from \( M \) and adding \( B \) to \( M \) we would obtain an \( n \times m \) simple matrix containing less 1’s than \( M \), a contradiction. \( \square \)

To prove Theorems 5 and 7 we need a lemma of Katona, which appears as Step C in the proof of Theorem 6 in [2].

**Lemma 9** (Katona) Let \( n \) and \( b \) be positive integers, \( b \leq n \). Let furthermore \( c \) be a positive integer satisfying \( c \leq \binom{n}{b+1} \). Then there exists an \( n \times c \) quasi-admissible matrix \( M(n, b, c) \), where every column has weight exactly \( b \).

Now we prove Theorem 5, from which Theorem 7 (by Lemma 8) immediately follows.

**Proof of Theorem 5:** By Theorem 4 and Claim 3 it suffices to show that \( n'(m, k) \leq n^*(m, k) \). For this, it is enough to show that if there exists an \( n \times m \) simple matrix \( M \) containing at most \( kn \) 1’s, then there exists an \( n \times m \) simple matrix \( M' \), where every row has weight at most \( k \). Let \( M \) be a simple \( n \times m \)
matrix containing at most \(kn\) 1’s, such that the number of 1’s is minimum. We have seen in the previous proof that for some \(j < n\) every column of \(M\) has weight at most \(j + 1\) and every column of weight at most \(j\) appears in \(M\). Now let us delete the columns of weight \(j + 1\) from \(M\) and add the columns of \(M(n, j + 1, m - \sum_{i=0}^{j} \binom{n}{i})\) to \(M\). The matrix \(M'\) obtained in this way is obviously an \(n \times m\) simple, quasi-admissible matrix containing the same number of 1’s as \(M\), which is at most \(kn\). Therefore (since \(M'\) is quasi-admissible), every row of \(M'\) has weight at most \(k\), which finishes the proof. \(\Box\)

References


