Beacon Deployment for Unambiguous Positioning

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***Abstract*—Instant and precise localization of a mobile user is fundamental for supporting various sophisticated indoor location-aware services. This study focuses on achieving unambiguous user positioning using practical Bluetooth Low Energy (BLE) beacons with multiple discrete power levels. By receiving the beacon coverage status from a user's device, the cloud server can unambiguously pinpoint the user's location and react correspondingly. We firstly define the problem of Beacon Deployment for Positioning (BDP) and provide several theoretic bounds on the number of required beacons in order to gain sufficient understanding on its performance behaviour. The BDP problem is further formulated into an integer linear program (ILP) and solved in extensive case studies. We claim that this is the first systematic and in-depth research on beacon deployment for unambiguous user positioning. Our analysis and experiment results show that the proposed solution takes only 2-8 times less beacons compared to that by the commonly adopted naive approach.**

***Index Terms*—Indoor Positioning, BLE beacons, Localization**

I. INTRODUCTION

INSTANT acquisition of precise position of each mobile user has long been considered essential for supporting various sophisticated indoor location-aware services. Since the user position information may not be available via Global Positioning System (GPS) due to the indoor shadowing effect, many alternatives based on various technologies have been reported, including radio (WiFi, ZigBee [1], [2], Bluetooth [3], [4], Bluetooth Low Energy [5]–[11], FM [12], and RFID [13]), optical (infrared, laser scanning systems [14] and camera-based [15]), audio (sound [15], ultrasound [16]), magnetic fields [17], barometer [18] and Inertial Navigation Systems (INS) [19].

According to [20], these indoor positioning techniques can be generally classified as anchor-based and non-anchor based solutions. With the anchor-based solution, anchor nodes are installed in prior and periodically broadcast their SSIDs, while the mobile nodes can determine their locations with the received SSIDs. If incorporating with advanced positioning algorithms (e.g. centroid [21], triangulation [22], and/or trilateration [23]), anchor-based solution can achieve better positioning granularity through manipulating various ranging information: received signal strength information (RSSI), time-of-arrival (ToA), time-difference-of-arrival (TDoA), and the angle-of-arrival (AoA). As an example of the anchor-based solution, the cell-based method [11] determines a probable *area* the user resides based on the visibility of the anchor nodes. For another, Nokia initialized the HAIP (High Accuracy Indoor Positioning) [24] project, where directional Bluetooth Low Energy (BLE) beacons can achieve 0.3m position accuracy for indoor users, at the expense of extra equipment and accurate installation in the training phase.

On the other hand, non-anchor-based solutions infer a user's location from reading the sensors on the mobile device. Odometry-based and fingerprinting is the most well known non-anchor based solutions. Odometry-based solutions use motion sensors and a map to determine the position change relative to the initial location. For fingerprinting, the fingerprint (readings from sensors) of interested locations will be created in the training phase. Subsequently, user location is identified by comparing current fingerprint with the trained database.

An example of non-anchor-based solution is the indoor positioning system (IPS) by Google [25]. It is characterized by a traditional site survey process as an offline training phase in order to build up a radio fingerprint database. The radio fingerprints can be by way of the sensed signal strength from multiple local-area or near-field access points, such as Wi-Fi, Bluetooth, and RFID. With the fingerprint database, the IPS can answer a user's location query according to the measured radio fingerprints by retrieving the matched fingerprints from its fingerprint database.

This study investigates unambiguous user positioning via the collaboration of a set of BLE beacons in the area of interest (AOI). As a cell-based method, our solution does not require a precise signal measurement and phase detection, but instead the mobile device only needs to determine whether it is in the radiation range of each beacon or not. In other words, the server localizes each mobile device simply by the SSIDs of the beacons successfully decoded and obtained at the mobile device, thus applicable to any type of BLE beacons and related protocols/SDKs, such as Apple's iBeacon, EddyStone beacon, Paypal Beacons, and Indoor.rs [26], etc. Different from other literature, our solution firstly supports multi-power-level BLE beacons and dramatically reduces the number of beacons needed by categorizing TPs into Shared Information Test Position Groups (SIPGs). We envision such an approach will be a norm in the future metropolitan and indoor positioning systems where low-cost sensors/beacons are massively deployed and analyzed via powerful cloud computing facilities.

We claim that the paper serves as the first study that formulates the beacon deployment task for unambiguous user positioning into a mathematical problem, namely Beacon Deployment for Positioning (BDP), with its goal as to ensure any two Test Positions (TPs) in different SIPGs being covered by distinct sets of beacons. Distinguished from any reported research, the proposed system model and formulated Integer Linear Program (ILP) are general to practical scenarios by

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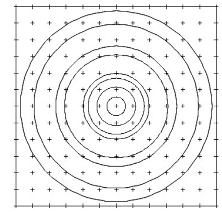
considering the fact that the effective radiation range of a BLE beacon could take any irregular shape. Based on the formulated problem, the paper provides extensive and in-depth analysis on the number of required beacons via geometry and coding theories.

The rest of the paper is organized as follows. We will firstly review the prior research of BLE based positioning systems in Section II. Section III presents the system model and the proposed beacon allocation problem. Section IV analyzes the proposed problem, and Section V presents the corresponding integer linear program (ILP), which is solved and the results are presented in Section VI. The paper is concluded in Section VII.

lev	pwr (dBm)	$rssi$ (dBm)
0	-30	-91
1	-20	-81
2	-16	-76
3	-12	-74
4	-8	-68
5	-4	-66
6	0	-62
7	4	-60

(a) Parameters for each power level



(b) The radiation pattern

Fig. 1. Power-levels of a beacon [30]

II. Related Work

[11] describes a cell-based positioning method using Bluetooth beacons capable of modelling complex reception characteristics and provides a greedy heuristic. Although the paper defined the problem of mono-power-level beacon deployment, it lacks any in-depth analysis and implementation. [27] leverages the capture effect by assuming only the strongest signal can be detected in a collision, in order to increase the lifetime of mobile nodes by three orders of magnitude and be more resilient to external interference. The position is determined merely by the coordinates of the BLE beacons without RSSI information. A platform using BLE beacons and ultrasound is introduced [16] by taking advantage of the difference in propagation characteristics between ultrasound and radio. With a prior calibration process that automatically computes the room geometry and the precise beacon locations, the system can track a user's location.

In [10], a smartphone-based indoor navigation system with BLE beacons and visual-tags is introduced. A rough localization is achieved by BLE beacons, followed by an accurate navigation via decoding the visual-tags with cameras. A Bluetooth-based IPS built upon low-cost off-the-shelf beacons and mobile devices is introduced in [4], which enables rapid prototyping of room-level applications. [7] pinpoints the user location by a neural network fed with data from beacons, achieving comparable accuracy to other approaches. [8] achieves a tracking accuracy of of 0.27 meters with iBeacon using particle filtering method.

[28] introduces a crowd-sourcing localization system based on Wi-Fi scene analysis and Bluetooth beacons. The mobile nodes submit their Wi-Fi fingerprints both to a map server and the bluetooth beacons, which further disseminate the collected fingerprints to all the nearby mobile nodes for instantly room-level positioning. [17] introduces a method for automatically generating and updating the RF signal map in buildings while localizing the mobile device. The trajectory of a smartphone is obtained by synthesizing the sensor measurement (inertial and RF signal) using an adaptation of the GraphSLAM technique. The pedestrian dead reckoning estimates avoids the users to hold the smartphone in hand and can accommodate multiple users.

[21] offers a low-lost, easy-deploying and reconfigurable positioning system. User location is determined by computing the weighted average of RSSI's and choosing the closest beacon. Depending on the deployment configuration, an accuracy down to 0.97 meters is possible. [29] proposes two schemes for indoor positioning by Bluetooth beacons and a pedestrian dead reckoning (PDR) technique to provide meter-level positioning. A multi-threshold step detection algorithm is adopted to improve the positioning accuracy using PDR. In addition, a heading estimation method with real-time compensation is proposed, built upon a Kalman filter with map geometry information. Moreover, this paper has implemented two positioning approaches.

In summary, Table I compares the features of our proposed work with current literature.

III. Problem Formulation

A 2D Positioning System is enabled in the Area of Interest (AOI), where the *Test Position* (TP) at which the user can be precisely localized is marked by "$+$"; and a possible position for installing a beacon, namely *Candidate Position* (CP), is marked by "$\times$". In practice, multiple beacons of different power levels can be placed at a common location in the AOI.

According to the application, multiple TPs could fall into a common *Shared Information Test Position Group* (SIPG) if the same location aware information is desired at those TPs.

Next we introduce the *Signal Coverage Model* for beacon deployment, which is followed by the formal definition of the BDP problem and the solution representation.

A. Signal Coverage Model

In the proposed user positioning system the location is identified only according to the beacons seen in the radiation range. Without loss of generality, we assume the beacons can be any type and has V discrete power levels[1]. As a part of the input we define, whether a TP can be covered by a beacon at CP according to 1) Radiation Pattern Specification, 2) Measurements or 3) Log Loss Radio Propagation Model. It is stored in Q_{ij}^k which is a binary indicator defined as follows

$$Q_{ij}^k = \begin{cases} 1, & \text{if TP } i \text{ is covered by a beacon at CP } j \\ & \text{ with power-level } k \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

[1]To the best of our knowledge, all current commercial BLE beacons has a limited set of discrete power levels [26], [30].

TABLE I
OUR WORK VS. LITERATURE

	anchor-based	bcn#	power-level	differentiate	granuality	online algorithm complexity
our work	T	low	multi	SIPGs	coarse	low
cell-based [4], [11],collocal [20]	T	moderate	mono	TPs	coarse	moderate
centroid [21]	T	high	mono	TPs	coarse	moderate
trilateration [23],triangulation [22]	T	high	mono	TPs	fine	high
beacons + extra medium [10], [16]	T	moderate	mono	TPs	fine	moderate
odometry [8], [17], [29]	F	N/A	N/A	TPs	fine	high
fingerprinting [7], [25], [28]	F	N/A	N/A	TPs	fine	moderate

See also Fig. 2c as an example for the radiation patterns of a beacon at a CP for each power level. Ideally, the *Radiation Pattern Specification* is provided by the beacon's manufacturer, usually obtained through experiments conducted in typical application environments such as offices, stores etc. Note that the radiation pattern can be different for each CP due to shadowing by any possible barrier in the radiation range of the beacon. To get more precise radiation patterns, case-by-case measurements should be performed.

Alternatively, the well-known Log Loss Radio Propagation model [31] can be adopted, where the beacon's radiation pattern will be a circle, as shown in (2):

$$PL(d) = PL(d_0) + 10\alpha \log_{10} \frac{d}{d_0} \tag{2}$$

where $PL(d)$ is the power loss in dB at d meters away, $PL(d_0)$ is the reference power loss at d_0 meters, and α is the attenuation factor.

For BLE beacons, RSSI values are provided by manufacturers [26], [30] for each power-level in related standards or data sheets, which represents the receiver side signal strength in dBm at $d_0 = 1$ meter away from the beacon. By definition, $rssi^k = p_t^k - PL^k(1)$, where $rssi^k$, p_t^k and $PL^k(1)$ are the RSSI, the transmission power of the beacon, and the first meter power loss at power-level $k \in \{0, 1, \dots, V-1\}$ respectively. If the receiver sensitivity is θ and the maximum distance a beacon at power-level k can effectively cover is d^k, then the maximum acceptable power loss from the beacon $PL^k(d) = p_t^k - \theta$ and $PL^k(d) - PL^k(1) = rssi^k - \theta$. From (2), $d^k = 10^{\frac{rssi^k - \theta}{10\alpha}}$ meters.

Thus Q_{ij}^k simply tests whether $D_{ij} \leq d^k$ where D_{ij} denotes the distance between CP i and TP j. Fig. 1b illustrates the circular radiation patterns when $\alpha = 3$ for the power-levels defined in Fig. 1a.

B. The Problem Definition

In this study, the problem of Beacon Deployment for Positioning (BDP) is defined with its goal of minimizing the number of beacons deployed at the CPs such that two TPs in different SIPGs can be differentiated through beacon coverage while the positioning delay and the power consumption of each beacon are constrained.

The inputs to the BDP problem includes the set of TPs (denoted by $\mathbb{T}$), the set of SIPGs (denoted by $\mathbb{G}$), the set of CPs (denoted by $\mathbb{C}$), the number of discrete power levels (denoted by V), and the beacon ranges for each CP (denoted by Q_{ij}^k). As described previously, the beacon ranges are computed based on the attenuation factor (denoted as α), the receiver sensitivity of user devices (denoted as θ), the beacon specific parameters such as power-level settings. We assume there are at least two TPs and one CP, formally $|\mathbb{T}| \geq 2$ and $|\mathbb{C}| \geq 1$.

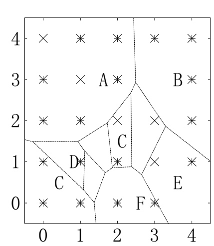
To limit the power consumption of each beacon the input also includes the maximum acceptable expected power for each beacon, a time slot constant for broadcasting interval, the transmission power for each power-level, and the range of broadcasting interval.

The outputs of the problem are the positions of the required beacons along with their power levels and broadcasting interval, which exclusively determine the set of beacons "seen" at each TP. The solution is represented by an *Area Code Table* (ACT). Each row in the ACT stores an *Area Code* (AC) corresponding to a unique set of beacon ids "visible" at the TPs listed in that row; each column is a *Beacon code* which indicates the SIPGs covered by the beacon of the column. Specifically, a binary bit in an AC is set to 1 if the corresponding beacon covers all TPs listed in that row while it is set to 0, otherwise. The last two rows of the ACT are used to store the power-level and broadcasting interval settings for each deployed beacon while the first column stands for the SIPG that each AC belongs to. Since all SIPGs must be differentiated, no ACs should be shared among them.

A special case of the BDP problem is the *Beacon Deployment for Unambiguous Positioning* (BDUP) problem where each SIPG has a single TP. Thus the solution of BDUP can serve as an upper bounds to the corresponding BDP.

C. Illustrative Example

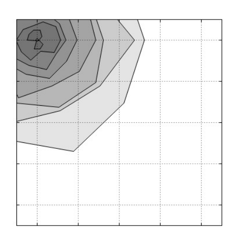
Fig. 2 exemplifies the proposed BDP problem where the CPs/TPs are shown in Fig. 2a and the AOI of $30 \cdot 30m^2$ is divided into sub-areas based on the SIPGs defined in Fig. 2b. Note that an SIPG may be the union of multiple physically



(a) TPs/CPs settings

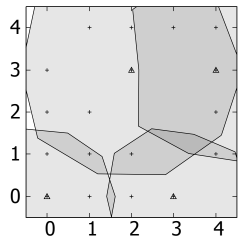
SIPG	TPs
A	(1,4)(2,4)(0,3)(2,3)(0,2)(1,2)
B	(3,4)(4,4)(4,3)(4,2)
C	(0,0)(0,1)(1,0)(2,1)
D	(1,1)
E	(4,1)
F	(2,0)(3,0)

(b) SIPGs to identify



(c) Radiation range of a beacon for each power level

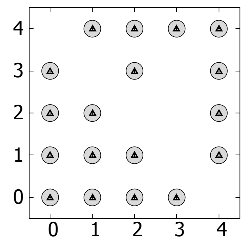
Fig. 2. A BDP problem



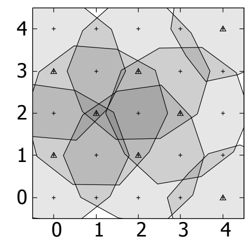
(a) The BDP solution

SIPG	CPs with beacons (0,0)	(3,0)	(2,3)	(4,3)	TPs
A	0	0	1	0	(0,2)(1,2)(0,3)(2,3)(1,4)(2,4)
B	0	0	0	1	(4,2)(4,3)
B	0	0	1	1	(3,4)(4,4)
C	1	0	0	0	(0,0)(1,0)(0,1)
C	0	1	1	0	(2,1)
D	1	0	1	0	(1,1)
E	0	1	0	1	(4,1)
F	0	1	0	0	(2,0)(3,0)
pwr(dBm)	-4	-4	4	0	
interv(sec)	0.1	0.1	0.51	0.2	

(b) The Area Code Table (ACT)



(c) The naive solution



(d) The BDUP solution

Fig. 3. Solutions to the problem defined by Fig. 2

non-adjacent sub-areas. As illustrated in Fig. 2c, a beacon at CP $(0, 4)$ has 8 randomly generated polygon radiation patterns according to specific power levels.

By solving the ILP in Section IV, an optimal solution to the problem defined in Fig. 2 is demonstrated in Fig. 3a and significantly outperforms the $O(|\mathbb{T}|)$ naive solution, commonly adopted by iBeacon etc., where each TP is equipped with a beacon (See Fig. 3c). Each deployed beacon is marked by "$\triangle$" while its radiation pattern is shown by the corresponding shaded area. Fig. 3b presents the resultant ACT. In contrast, Fig. 3d shows a corresponding BDUP solution when all positions are TPs/CPs.

The generated ACT is kept in the server and the SIPG is identified when a set of beacon IDs (i.e. AC) is submitted by a mobile device. The server instantly identifies the SIPG with which the mobile device is associated by a table look-up.

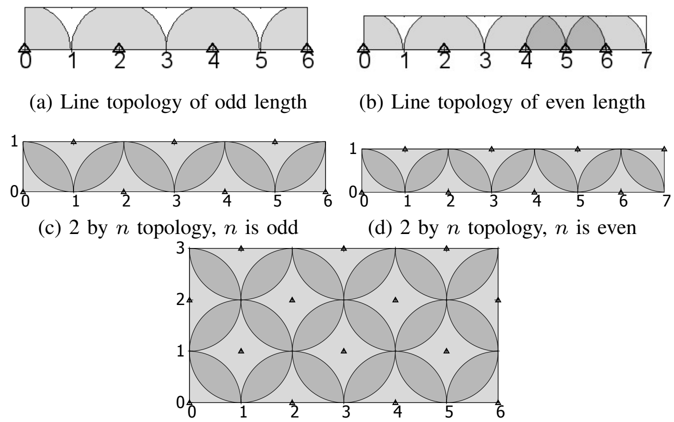
IV. Problem Feasibility

Next we give bounds on the number of beacons without considering the positioning delay and the power consumption. First we give simple general lower bounds. It is followed by a general lower bound based on the geometry of the beacon range. Next we formulate lower bounds depending on the maximal number of TPs a beacon can cover. Finally, we give several upper bounds on the number of beacons.

Definition 1: A solution is feasible to the BDP problem if and only if

1) for any two TPs in different SIPGs, there exists a beacon covering exactly one of them;
2) all TPs are covered.

Lemma 1: Given $\mathbb{C} \supseteq \mathbb{T}$ and let no more than one TP be covered by a beacon with the minimum power level. There's a trivial solution for BDUP by installing a beacon at each CP $\in \mathbb{C}$.



(a) Line topology of odd length

(b) Line topology of even length

(c) 2 by n topology, n is odd

(d) 2 by n topology, n is even

(e) m by n topology

Fig. 4. BDUP solutions: simple beacon patterns

Proof: Directly follows from Definition 1 since each TP will be assigned a unique area code of weight 1. ■

Lemma 2: The minimum beacon number for BDUP is $\lceil \frac{|\mathbb{T}|+1}{2} \rceil$ in a line topology of $|\mathbb{T}|$ TPs (i.e. the TPs are in a row as shown on Fig. 4a). This bound is tight.

Proof: 1) At least $\lceil \frac{|\mathbb{T}|+1}{2} \rceil$ are needed. Consider a line with $|\mathbb{T}|$ TPs, $|\mathbb{T}| + 1$ boundaries are needed to cut these TPs. Note that adding a beacon at most contributes 2 boundaries, hence at least $\lceil \frac{|\mathbb{T}|+1}{2} \rceil$ beacons are required to fully differentiate the TPs.

2) there exists a solution with $\lceil \frac{|\mathbb{T}|+1}{2} \rceil$ beacons. A 3-hop chaining pattern can be applied. When $|\mathbb{T}|$ is odd, a perfect chain is formed (see Fig. 4a); when $|\mathbb{T}|$ is even, an additional cycle is added near the end (see Fig. 4b). Note that each TP is uniquely covered by $1 - 3$ beacons. ■

A. Bounds based on beacon coverage geometry

Theorem 1: The number of beacons for BDUP is at least

$$\sqrt{\frac{2|\mathbb{T}|}{\mu} + \frac{1}{4} - \frac{2}{\mu}} + \frac{1}{2} \tag{3}$$

where μ is the maximum number of times the boundary of two beacons can intersect.

Proof: By viewing the AOI as a plane and the beacons as geometric shapes, we will show that b shapes can divide the plane into at most $1 + \mu\frac{b(b-1)}{2}$ regions, where μ is the maximum number of times the boundary of two shapes can intersect.

The proof is inductive inspired by the argument in [32, page 243]. Let $r(b)$ denote the number of regions into which the plane can be divided by b shapes excluding the outside region where a TP would have an all zero area code. A single shape can divide the plane into an inside and outside regions, i.e. $r(1) = 1$. By adding the $b + 1$-th shape, the number of regions of the plane increases by the number of existing regions that the new shape intersects. The new shape has at most μ intersections with each of the previous shapes, thus in total it has $n\mu$ intersections, each of which divides one of the original regions into two. Hence $r(b+1) \leq r(b) + b\mu$. An explicit formula for $r(b)$ is as follows:

$$r(b) \leq r(b-1) + (b-1)\mu \leq r(b-2) + (b-2)\mu + (b-1)\mu \leq \cdots \leq r(1) + (1 + \cdots + b - 1)\mu \leq 1 + \frac{b(b-1)}{2}\mu \quad (4)$$

Therefore, the minimum number of required beacons b, must satisfy:

$$1 + \frac{b(b-1)}{2}\mu = \frac{\mu}{2}b^2 - \frac{\mu}{2}b + 1 \geq |\mathbb{T}| \Rightarrow b \geq \sqrt{\frac{2|\mathbb{T}|}{\mu} + \frac{1}{4} - \frac{2}{\mu}} + \frac{1}{2} \quad (5)$$

■

Corollary 1: With every beacon being in a shape of cycle, the number of beacons for BDUP is at least $\sqrt{|\mathbb{T}| - \frac{3}{4}} + \frac{1}{2}$.

Proof: Two cycles can intersect at most two points, thus $\mu = 2$ in Theorem 1. ■

Corollary 2: With every beacon being in a shape of ellipse, the number of beacons for BDUP is at least $\sqrt{2|\mathbb{T}| - \frac{1}{4}} + \frac{1}{2}$.

Proof: Two elips can intersect at most two points, thus $\mu = 4$ in Theorem 1. ■

Corollary 3: If every beacon has a shape of polygon with at most x corners, the number of beacons needed for BDUP is at least $\sqrt{\frac{2|\mathbb{T}|}{x^2} + \frac{1}{4} - \frac{2}{x^2}} + \frac{1}{2}$.

Proof: Two lines can intersect in at most one point. Thus two polygons with at most x lines can intersect at most x^2 points, thus $\mu = x^2$ in Theorem 1. ■

B. Bounds based on the maximum number of TPs a beacon can cover

The problem of identifying items via tests have been deeply studied as separating systems. The version where the tests have restrictions on their size was first raised by Rényi in 1961; and the first and most important results are due to Katona (1966).

Theorem 2 ([33]): Let Φ denote the maximum number of TPs each beacon covers. The number of beacons for BDUP is at least b satisfying the following inequalities. Find the least number b, for which there exist natural numbers $j \leq b - 1$ and $a < \binom{b}{j+1}$, such that

$$\sum_{i=0}^{j} i \cdot \binom{b}{i} + a(j+1) \quad \leq \quad \Phi b, \quad (6)$$

$$\sum_{i=0}^{j} \binom{b}{i} + a \quad = \quad |\mathbb{T}| + 1. \quad (7)$$

There is no closed formula known for the above bound, but $\lceil \log_2(|\mathbb{T}|+1) \rceil$ is the information theoretical lower bound and it is known that

Theorem 3 ([33]): Let Φ denotes the maximum number of TPs a beacon can cover. The number of beacons for BDUP is at least

$$\frac{|\mathbb{T}|+1}{\Phi} \cdot \frac{\ln(|\mathbb{T}|+1)}{1 + \ln\frac{|\mathbb{T}|+1}{\Phi}} \quad (8)$$

C. Constructions for beacons with circular pattern

Finally, upper bounds on BDUP for beacons with circular radiation patterns are introduced. We assume the distance between two adjacent TPs in the grid AOI is 1 unit, and let r denote the maximum radius of the beacons.

Lemma 3: Consider a 2 by n grid AOI. The upper bound on the beacon number for BDUP is n for $r = 1$.

Proof: Fig. 4c and 4d demonstrate the construction patterns for odd and even n. Note that each TP is uniquely covered by $1, 2$ or 4 beacons. ■

Corollary 4: For a m by n grid AOI (WLOG, $m \geq n$), BDUP can be achieved by $\lfloor \frac{m}{2} \rfloor n + (m \mod 2)\lceil \frac{n}{2} \rceil$ beacons for $r = 1$.

Proof: A solution to the m by n topology can be constructed by keeping on applying the 2 rows patterns (See Fig. 4c and 4d). If there's one row remaining, additionally $\lceil \frac{n}{2} \rceil$ beacons are needed.

Fig. 4e illustrates the construction when $m = 4$ and $n = 7$. Note that for any row (e.g. row 1), the beacons on it and its adjacent rows (i.e. rows $0-2$) separate it from the remaining rows. Plus, by Theorem 3, the TPs in current row are differentiated from those on its adjacent rows. Thus, any row is uniquely coded and all TPs can be identified.

■

Theorem 4: For a m by n grid AOI where $m, n \in [3, 10]$ and $\lceil \frac{m}{2} \rceil + 1 \leq n$, no more than $8R+4$ beacons of radius $r = \sqrt{2}R$ are needed to achieve BDUP, where $R = \frac{\lceil \frac{m}{2} \rceil - (\lceil \frac{m}{2} \rceil \mod 2)}{2}$.

Proof: As shown in Fig. 5a, when $m \leq 10$, $R \leq 2$ and any beacon with radius $\sqrt{2}R$ covers $(2R+1)^2$ TPs totally as its inner square does. Thus, such a beacon can be treated as a square which partially cuts the rows and columns simultaneously. Due to our choice of R, any beacon can cover at least half of the TPs per row/column, thus with two such beacons any row/column can be covered.

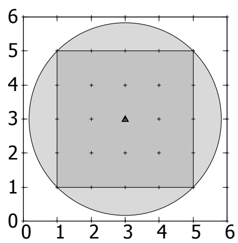
Based on the above observations, the pattern shown in Fig. 5b-5c can be applied to differentiate the TPs. Basically, the top and bottom array of beacons are used to cut the columns while the left and right array of beacons are used to cut the rows using the pattern introduced in Lemma 4 of the Appendix.

Due to choice of R, $4R + 1 = 2(\lceil \frac{m}{2} \rceil - (\lceil \frac{m}{2} \rceil \mod 2)) + 1 \geq 2(\lceil \frac{m}{2} \rceil - 1) + 1 \geq m - 1$, thereby at most $2R + 2$ beacons are needed for differentiating a row/column. Since $2R+2 = \lceil \frac{m}{2} \rceil - (\lceil \frac{m}{2} \rceil \mod 2) + 2 \leq \frac{m+1}{2} + 2$, when $m \geq 3$ no more than m beacons are needed (if $m = 3, 4$, exactly 3 and 4 beacons are needed while $m \geq 5$ implies $2R+2 \leq m$), thus we can always find a feasible solution. Since 4 beacons are shared as shown in Fig. 5b and 5c, no more than $4(2R+2)-4$ beacons are needed by Lemma 4. ■

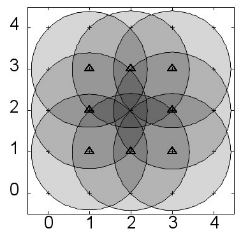
Note that the above theorem 3-optimal when $m = n$. By Corollary 1, at least $m+1$ beacons are required and we have $\frac{4(\lceil \frac{m}{2} \rceil - (\lceil \frac{m}{2} \rceil \mod 2)+1)}{m+1} \leq \frac{4(\frac{m+1}{2}+1)}{m+1} = 2 + \frac{4}{m+1}$. When $m = 3$, the worst ratio is 3, thus the solution is 3-optimal.

Corollary 5: For a m by n AOI (m, n are multiples of 10), BDUP is achievable with $\frac{1}{5}mn$ beacons of radius $2\sqrt{2}$.

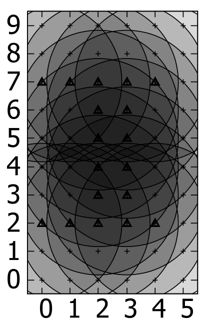
Proof: Note that each square area with 10 by 10 TPs can be differentiated by 20 beacons of radius $2\sqrt{2}$ by Theorem 4,



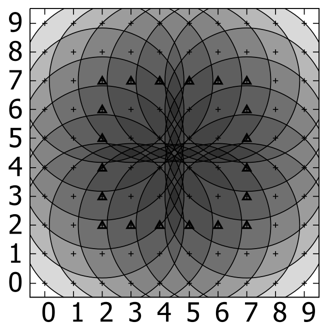
(a) TPs covered by a beacon of radius $r = 2\sqrt{2}$



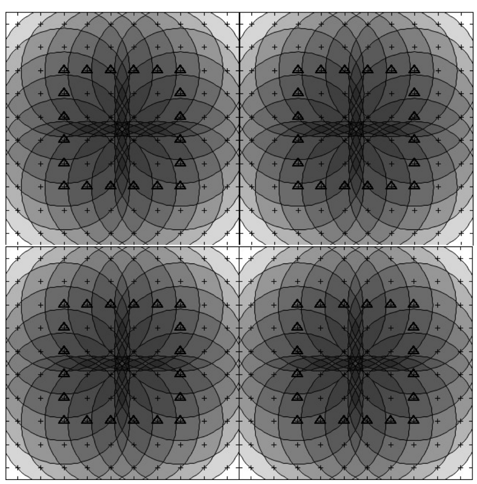
(b) 5 by 5 solution



(c) 10 by 6 solution



(d) Optimal 10 by 10 solution



(e) Solution if m, n are multiples of 10.

Fig. 5. BDUP solution: m by n constructions

as illustrated in Fig. 5d. Since each square area will not cover any other TPs outside its boundary, the TPs in each square are independently coded and can be pieced together, thus proved the BDUP by using $\frac{1}{5}mn$ beacons. Fig. 5e shows the case when $m = n = 20$. ∎

Note that the above idea can be generalized when m and n are not multiples of 10. For example, for a 12 by 12 AOI, BDUP can be achieved with 42 beacons by placing a solution of 10 by 10 as on Fig. 5d with 20 beacons on the top right corner, and a solution of 12 by 2 on the left side from Lemma 3 (Fig. 4c) with 12 beacon and a solution 2 by 10 on the bottom right corner with 10 beacons. Similarly, for a 15 by 15 AOI, BDUP can be achieved with 60 beacons by placing a solution 10 by 10 with 20 beacons on the top right corner surrounded by five 5 by 5 solutions of Fig. 5b each with 8 beacons.

V. ILP FORMULATION FOR BDP

In this section, common notations used by the proposed ILP are listed. Without confusion, all greek or upper-case notations are given or precomputed constants, mathbold symbols represent given sets, while the lower-case alphabetic notations are the decision variables.

First we introduce our basic which similar to Section IV ignores the positioning delay and the power consumption.

A. Basic ILP formulation

Input Parameters:

$\mathbb{T}/\mathbb{C}/\mathbb{G}$: set of TPs/CPs/SIPGs (note that each CP can be placed with a single beacon, while a location can be mapped by multiple CPs).

V: number of discrete power levels.

Q^k_{ij}: indicates whether a beacon at CP i at power-level k can cover TP j.

$\mathbb{C}_i$: set of CPs can cover TP i with maximum power-level.

B_{min}: a lower bound on the number of beacons.

Decision Variables:

$v_{ij} \in \{0,1\}$: If it is 1 install a beacon at CP i with power level j; 0 otherwise.

$q_{ij} \in \{0,1\}$: It takes 1 when a beacon at CP i covers TP j; it takes 0 otherwise.

$x^k_{ij} \in \{0,1\}$: Used to help computing the XOR value of two binary variables, formally $x^k_{ij} = q_{ki} \cdot q_{kj}$.

The BDP problem can be formulated as follows:

$$\text{(BDP)} \min \sum_{i \in \mathbb{C}} \sum_{j=0}^{V-1} v_{ij}, \tag{9}$$

Subject to:

$$\sum_{j=0}^{V-1} v_{ij} \leq 1, \qquad \forall i \in \mathbb{C} \tag{10}$$

$$\sum_{i \in \mathbb{C}} \sum_{j=0}^{V-1} v_{ij} \geq B_{min} \tag{11}$$

$$q_{ij} = \sum_{k=0}^{V-1} Q^k_{ij} v_{ik}, \qquad \forall i \in \mathbb{C} \tag{12}$$

$$\sum_{j \in \mathbb{C}_i} q_{ji} \geq 1, \qquad \forall i \in \mathbb{T} \tag{13}$$

$$\sum_{k \in \mathbb{C}_i \setminus \mathbb{C}_j} q_{ki} + \sum_{k \in \mathbb{C}_j \setminus \mathbb{C}_i} q_{kj} + \sum_{k \in \mathbb{C}_i \cap \mathbb{C}_j} (q_{ki} + q_{kj} - 2x^k_{ij}) \geq 1, \quad \forall i,j \in \mathbb{T}, \notin \text{same SIPG} \tag{14}$$

$$2x^k_{ij} \leq q_{ki} + q_{kj} \leq x^k_{ij} + 1, \quad \forall i,j \in \mathbb{T}, \notin \text{same SIPG}, k \in \mathbb{C}_i \cap \mathbb{C}_j \tag{15}$$

The objective function (9) minimized the total number of required beacons, where $\sum_{j=0}^{V-1} v_{ij}$ is one if a beacon is placed at CP i.

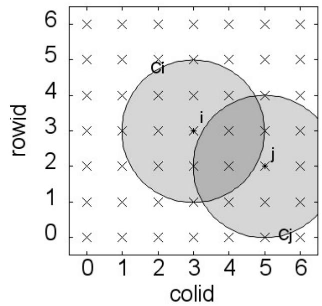


Fig. 6. Area Code Demo

The constraint (10) illustrates: if a beacon is installed at CP i, only one power-level can be selected[2] (i.e. only one $v_{ij} = 1$); otherwise, $v_{ij} = 0,\ \forall j = \{0, 1, \ldots, V-1\}$. The lower bounds of Thm.1, 2 and 3 is added as a constraint (11) specifying that at least B_{min} beacons are in the feasible solution.

As shown in (12), to decide whether a beacon at CP i can cover TP j, we need to compute q_{ij} based on the precomputed values Q^k_{ij}'s at each power-level. When the power radiation patterns are circular, only a portion of the TPs need to be determined for CP i due to symmetry, which can speed up the ILP computation. The constraint (13) ensures each TP to be covered by some beacon(s).

Any two TPs in different SIPGs should be covered by a different set of beacons, which is achieved by the constraints (14) and (15). The sufficient conditions of differentiating two TPs i and j are as follows:

1) either some beacon(s) are installed at $\mathbb{C}_i \setminus \mathbb{C}_j$, such that there is a beacon only covers i;
2) or installed at $\mathbb{C}_j \setminus \mathbb{C}_i$, such that there is a beacon only covers j;
3) or installed at $\mathbb{C}_i \cap \mathbb{C}_j$ without covering both TPs.

This is also demonstrated in Fig. 6. The three conditions are handled by the three terms on the LHS of (14), respectively. For condition 3), a beacon installed at $\mathbb{C}_i \cap \mathbb{C}_j$ may or may not cover both TPs depending on the power-level selected. To test whether the beacon covers both TPs, we need to compute $\sum_{k\in\mathbb{C}_i\cap\mathbb{C}_j} q_{ki} \oplus q_{kj}$, which is translated into the third term in (14) together with (15).

B. Considering the constraint on transmission power and adding the positioning delay as an objective

In this subsection we extend the objective function (9) with a target of shortening the positioning delay and add a constraint on maximal transmission power. First we introduce a new input parameters

P_{max}: maximum allowed expected power (mw) of a beacon.

τ: time slot constant for broadcasting interval ($= 625\mu s$).

ω_i: transmission power[3] (mw) for power-level i.

[2]Alternatively, we can define the set $\{v_{ij}, \forall j\}$ as a Special Ordered Set of Type 1 (SoS1) for each TP i in CPLEX. Specifically, the SoS1 weight for v_{ij} is set to $rssi^j - rssi^0$ where $rssi^j$ refers to the RSSI value for the j-th power level as shown in Fig. 1a.

[3]Note that $\omega_i = 10^{tpwr[i]/10}$ where $tpwr[i]$ is the given transmission power in dBm for power-level i.

$S_{min(max)}$: the broadcasting interval of a beacon.

Besides we introduce new working variables:

$s_i \in \{0, 1, \ldots S_{max}, \}$: Broadcasting interval for a beacon at CP i in time slots. If no beacon is at i, then $s_i = 0$.

$a_i \in \{0, 1, \ldots, S_{max}\}$: The maximum delay at TP i. $a_i = \max_{j\in\mathbb{C}_i}\{s_j\}$. i.e. the maximum broadcasting interval in the surrounding area.

$p_i \in \mathbb{R}^+_0$: Represents the transmission power at CP i.

The new objective function would be:

$$\text{(BDP)}\min \sum_{i\in\mathbb{C}}\sum_{j=0}^{V-1} v_{ij}, + \frac{1}{S_{max}|\mathbb{T}|+1}\sum_{i\in\mathbb{T}} a_i \tag{16}$$

Note that the second term in (9) is less than one such that it will not affect the beacon number required in the solution. The constraints are Eq. (10)-(15) along with

$$p_i = \sum_{j=0}^{V-1} \omega_j v_{ij}, \qquad \forall i \in \mathbb{C} \tag{17}$$

$$p_i \le P_{max}\tau s_i, \qquad \forall i \in \mathbb{C} \tag{18}$$

$$S_{min}\sum_{j=0}^{V-1} v_{ij} \le s_i, \qquad \forall i \in \mathbb{C} \tag{19}$$

$$S_{max}\sum_{j=0}^{V-1} v_{ij} \ge s_i, \qquad \forall i \in \mathbb{C} \tag{20}$$

$$s_j \le a_i, \qquad \forall i \in \mathbb{T}, j \in \mathbb{C}_i \tag{21}$$

The constraints (17) computes the transmission power at CP i respectively. The expected power consumption at any beacon should not exceed the given threshold, which is stated in (18). If P_{exp} or S_{max} decreases, possibly less power-levels are valid for the given problem.

The constraint (19) and (20) specifies the valid broadcasting interval range for a beacon. The constraints (21) computes the estimated positioning delay at TP i. Note that $a_i = \max\{s_j\}, j \in \mathbb{C}_i$, which is formulated using the big-M method.

VI. Case Studies

In this section, case studies are conducted to validate the proposed BDUP problem and the ILP. To simulate practical BDUP applications, various scenarios with different attenuation factors and TPs/CPs settings are considered. For all cases, the total area of the AOI is $60 \cdot 60m^2$ where TPs/CPs are evenly distributed in it. The power-level settings of the beacons are listed in Fig. 1a, which simulates the commercial estimote BLE beacons. The acceptable positioning delay is from 0.1 to 2 secs, the receiver sensitivity of any user's device is set to -97dBm to simulate an ordinary mobile phone, and the maximum acceptable expected power-consumption for a beacon is 5mw.

In our experiments, attenuation factors from 3 to 5 are chosen to model common indoor environment while sparse to dense TPs/CPs settings are examined to study the performance

TABLE II
SIMULATION DETAILS: 60 BY 60 METERS, $\theta = -97$DBM, MAXIMUM DELAY $= 2secs$, $\bar{p}_{max} = 5mw$. THE SAVE REPRESENTS THE NUMBER OF BEACONS COMPARED TO THE NAIVE SOLUTION, COMMONLY ADOPTED BY IBEACON ETC.

Input				lower bound on bcn#			upper bound		ILP				
α	$\lvert\mathbb{T}\rvert$	Φ	r	Thm.1	Thm.2	Thm.3	Cor 4	Cor 5	bcn#	**save**	gap%	time(sec)	rows, cols, nzrs
3	20×20	101	7.1	21	12	11	200	80	50	8x	40.41	221568	$2{\cdot}10^5$, $1{\cdot}10^5$, $7{\cdot}10^6$
3	15×15	61	4.9	16	11	9	113	(60)	35	6.4x	31.40	31243.5	$1{\cdot}10^5$, $4{\cdot}10^4$, $2{\cdot}10^6$
3	12×12	37	3.7	13	10	9	72	(42)	27	5.3x	23.57	24138	$4{\cdot}10^4$, $1{\cdot}10^4$, $4{\cdot}10^5$
3	10×10	25	2.9	11	10	8	50	20	20	5x	0	48.93	$1{\cdot}10^4$, $6{\cdot}10^3$, $1{\cdot}10^5$
4	20×20	21	2.3	21	37	29	200		89	4.5x	29.91	22198.5	$7{\cdot}10^4$, $3{\cdot}10^4$, $6{\cdot}10^5$
4	15×15	13	2.1	16	32	25	113		51	4.4x	24.00	2723	$3{\cdot}10^4$, $1{\cdot}10^4$, $2{\cdot}10^5$
4	12×12	9	1.7	13	29	22	72		40	3.6x	11.00	11336	$9{\cdot}10^3$, $5{\cdot}10^3$, $5{\cdot}10^4$
4	10×10	5	1.1	11	33	24	50		39	2.6x	15.38	2115	$5{\cdot}10^3$, $3{\cdot}10^3$, $2{\cdot}10^4$
5	20×20	9	1.7	21	80	56	200		116	3.4x	25.35	6252	$3{\cdot}10^4$, $1{\cdot}10^4$, $1{\cdot}10^5$
5	15×15	5	1.1	16	75	51	113		89	2.5x	15.73	658.5	$1{\cdot}10^4$, $7{\cdot}10^3$, $4{\cdot}10^4$
5	12×12	5	1.1	13	48	34	72		56	2.6x	14.29	202.5	$5{\cdot}10^3$, $3{\cdot}10^3$, $2{\cdot}10^4$

and scalability of our proposed work under various positioning granularity and problem size.

Table II summarizes the results we obtained, where the Φ column represents the maximum number of TPs can be covered by a beacon and the r column denotes the corresponding maximum radius of the beacon. It also shows the upper and lower bounds sof Section IV and the details of the ILP. In our experiments the solution by ILP requires 2-8 times less beacons compared to the commonly adopted naive approach commonly adopted, like iBeacon, etc.

Fig. 7 examines the case that the radiation pattern is circular, which is applicable when the beacons are installed at the ceilings of an indoor environment with few obstacles. Table II summaries the results in Fig. 7 along with the scenario settings, where the LB column lists the analytical lower bounds obtained from Theorems 1, 2 and 3 while the last column indicates the size of the ILP problem. Each case is solved on CPLEX 12.6 with "mip cuts all" settings by using a server with 16G RAM and a 4 core 3.6GHz Intel CPU.

Fig. 7i plots the performance gap between the ILP under various CP/TP and attenuation factor settings based on Table II. The x-axis represents the number of TPs while the y-axis represents the corresponding optimal solution for the number of beacons required. As illustrated in Fig. 7i, the number of beacons grows almost linearly with respect to the number of TPs.

Fig. 7 demonstrates the optimal solutions under various scenarios where each beacon is drawn as a circle with its center marked by "$\triangle$" and its radius corresponds to the farthest grid distance it can cover. As shown in Fig. 7, each TP is covered by a unique set of beacons.

VII. CONCLUSIONS

In this paper, the problem of beacon deployment for unambiguous user positioning (BDP) was investigated. Based on the problem definition and feasibility, we theoretically proved a series of performance bounds on the number of required beacons, and formulated a novel ILP that jointly determines the beacon positions along with their power levels and broadcast intervals. We claim that this is the first complete research work on beacon deployment with multiple power levels and irregular radiation patterns. The ILP was solved and our concluded observations are given as follows:

1) the analytical bounds we derived are tight with the ILP results with about 20% of gap, which can serve as a viable guide in designing for larger instances;
2) with a larger power constraint (thus a larger radiation range), less beacons are required at the expense of shorter beacon lifetime and/or longer delay;
3) the number of required beacons goes approximately linearly with the increase of the number of TPs in an AOI
4) In our experiments the solution by ILP requires 2-8 times less beacons compared to the commonly adopted naive approach commonly adopted, like iBeacon, etc..

Our future research will be on development of effective heuristic algorithms in solving the BDP problem in order to gain more insights into the performance behaviour of the proposed approach under larger and irregular topologies.

APPENDIX A

Lemma 4: For a 1 by n AOI, BDUP is achievable with $2\lfloor r\rfloor+1$ beacons of radius r when $n \in [2\lfloor r\rfloor+1, 4\lfloor r\rfloor+1]$ and $n-2\lfloor r\rfloor$ such beacons when $n > 4\lfloor r\rfloor+1$.

Proof: By consecutively putting $2\lfloor r\rfloor+1$ beacons of radius r, any 1 by n AOI ($n \in [2\lfloor r\rfloor+1, 4\lfloor r\rfloor+1]$) can be differentiated using the pattern demonstrated in Fig. 8a(if $n < 4\lfloor r\rfloor+1$, the pattern should be shifted and truncated to fit n while always keeping the $2\lfloor r\rfloor+1$ beacons, see Fig. 8b).

When $n > 4\lfloor r\rfloor+1$, in addition to the $2\lfloor r\rfloor+1$ beacons, another $n-(4\lfloor r\rfloor+1)$ beacons are required at the end of the line AOI, and all TPs can be differentiated as Fig. 8c demonstrates.

As shown in Fig. 8, the pattern is coded as a block diagonal matrix. When $n = 4\lfloor r\rfloor+1$, a n by $2\lfloor r\rfloor+1$ matrix is used, as shown in Fig. 8d; when $n \in [2\lfloor r\rfloor+1, 4\lfloor r\rfloor+1)$ or $n > 4\lfloor r\rfloor+1$, the pattern should be shifted and truncated (See Fig. 8e) or appended to fit n (See Fig. 8f). ∎

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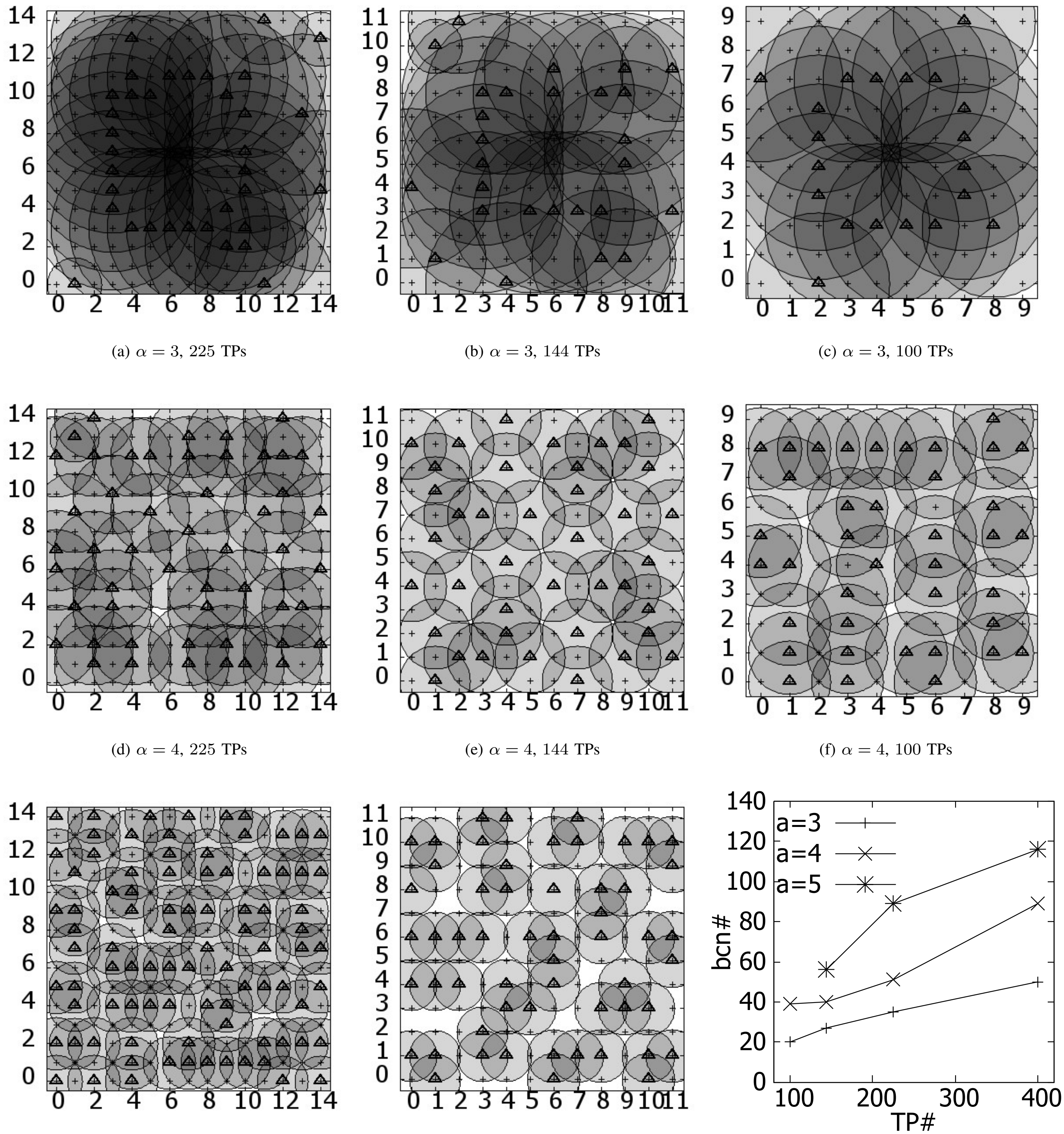

(a) $\alpha = 3$, 225 TPs (b) $\alpha = 3$, 144 TPs (c) $\alpha = 3$, 100 TPs

(d) $\alpha = 4$, 225 TPs (e) $\alpha = 4$, 144 TPs (f) $\alpha = 4$, 100 TPs

(g) $\alpha = 5$, 225 TPs (h) $\alpha = 5$, 144 TPs (i) ILP simulation Results

Fig. 7. Beacon ILP Deployment: $60 \cdot 60m^2$ meters, $\theta = -97$dBm, maximum delay $= 2secs$, $\bar{p}_{max} = 5mw$

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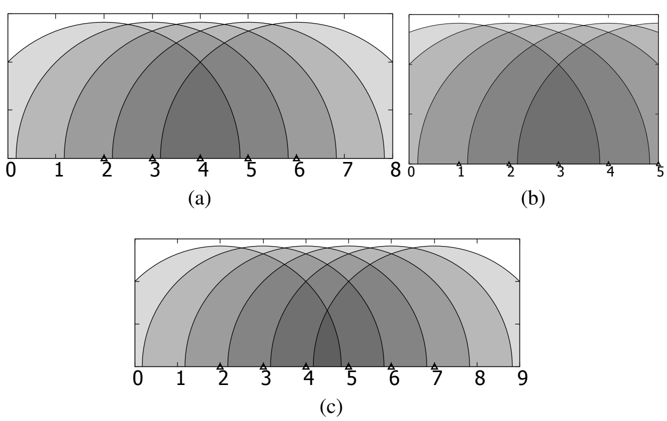

TP/bcn	23456
0	10000
1	11000
2	11100
3	11110
4	11111
5	01111
6	00111
7	00011
8	00001

(d)

TP/bcn	12345
0	10000
1	11000
2	11100
3	11110
4	01111
5	01111

(e)

TP/bcn	234567
0	100000
1	110000
2	111000
3	111100
4	111110
5	011111
6	001111
7	000111
8	000111
9	000011

(f)

Fig. 8. BDUP solution: 1 by n construction

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